

## Approximate F tests

In the factorial treatment design with three random factors, we have

$$E(MS_C) = \sigma^2 + r\sigma_{abc}^2 + rb\sigma_{ac}^2 + ra\sigma_{bc}^2 + rab\sigma_c^2$$

To test the null hypothesis  $H_0 : \sigma_c^2 = 0$ , we would like to have a mean square with expectation  $\sigma^2 + r\sigma_{abc}^2 + rb\sigma_{ac}^2 + ra\sigma_{bc}^2$ , but no single mean square satisfies this expression. However, we do have:

$$\begin{aligned} E(MS_{AC}) &= \sigma^2 + r\sigma_{abc}^2 + rb\sigma_{ac}^2 \text{ and} \\ E(MS_{BC}) &= \sigma^2 + r\sigma_{abc}^2 + ra\sigma_{bc}^2 \end{aligned}$$

which sum to yield:

$$E(MS_{AC}) + E(MS_{BC}) = 2\sigma^2 + 2r\sigma_{abc}^2 + rb\sigma_{ac}^2 + ra\sigma_{bc}^2,$$

which has too many  $\sigma^2 + r\sigma_{abc}^2$  terms. So if we subtract the three-way interaction term we have

$$\begin{aligned} E(MS_{AC}) + E(MS_{BC}) - E(MS_{ABC}) &= 2\sigma^2 + 2r\sigma_{abc}^2 + rb\sigma_{ac}^2 + ra\sigma_{bc}^2 - (\sigma^2 + r\sigma_{abc}^2) \\ &= \sigma^2 + r\sigma_{abc}^2 + rb\sigma_{ac}^2 + ra\sigma_{bc}^2 \end{aligned}$$

Thus we can use the F statistic

$$F = \frac{MS_C}{MS_{AC} + MS_{BC} - MS_{ABC}}$$

to test  $H_0 : \sigma_c^2 = 0$ . To approximate the distribution of this F statistic with an F distribution, we use Satterhwaite's approximation that a linear combination of mean squares,  $M = \sum a_i MS_i$ , has an approximate degrees of freedom of:

$$\nu = \frac{M^2}{\sum \frac{(a_i MS_i)^2}{\nu_i}}$$

Satterhwaite's result is used to obtain the degrees of freedom for the denominator of the F statistic. The F statistic above is commonly used for this test, but the text points out that there are advantages to instead using the F statistic of:

$$F = \frac{MS_C + MS_{ABC}}{MS_{AC} + MS_{BC}}$$

so that negative values cannot occur.