Use of Generalized Inverse Matrices for One-Way ANOVA

For most of the course we will assume that the $\mathbf{X}'\mathbf{X}$ matrix is nonsingular, so the least-squares solution is unique. Here we examine an approach that can be used when the $\mathbf{X}'\mathbf{X}$ matrix is singular. Suppose that we have data from two groups, with sample sizes n_1 and n_2, respectively, with total sample size $n = n_1 + n_2$. We use the overparametrized model,

$$Y_{ij} = \mu + \alpha_j + \epsilon_{ij},$$

so that the model matrix $\mathbf{X}$ has the following form:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix},$$

and the $\mathbf{X}'\mathbf{X}$ matrix is:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & n_1 & n_2 \\ n_1 & n_1 & 0 \\ n_2 & 0 & n_2 \end{bmatrix},$$

which only has rank 2.

A generalized inverse of an $m\,x\,n$ matrix $\mathbf{B}$ is defined to be any $n\,x\,m$ matrix $\mathbf{B}^-$ that satisfies

$$\mathbf{B}\mathbf{B}^-\mathbf{B} = \mathbf{B}.$$

If the model

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon$$

has an $\mathbf{X}'\mathbf{X}$ matrix that is singular then a generalized inverse can be used to obtain a least squares solution:

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y}.$$

This solution, however, is not unique. We will discuss one way to obtain a generalized inverse and use it for the problem above.