Yet More Matrix and Statistical Results

Result 3: Let $\mathbf{y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, let $\mathbf{C}$ be an m **x** n matrix, and $\mathbf{d}$ be an m **x** 1 vector. Then $\mathbf{Cy}+\mathbf{d} \sim N_m(\mathbf{C}\boldsymbol{\mu}+\mathbf{d}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')$.

Result 4: Suppose $\mathbf{y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and let $\mathbf{u} = \mathbf{Ay}$ and $\mathbf{v} = \mathbf{By}$.Then $\mathbf{u}$ and $\mathbf{v}$ are independent if and only if $\text{Cov}(\mathbf{u}, \mathbf{v}) = \mathbf{A}\boldsymbol{\Sigma}\mathbf{B}' = \mathbf{0}$.

To prove result 4 we will prove a theorem about marginal distributions of a multivariate normal distribution using moment generating functions. Recall that the moment generating function (mgf) of a random variable, denoted by $M_X(t)$ is

$$M_X(t) = E\, e^{tX},$$

provided that the expectation exists for t in some neighborhood of 0.

Theorem 1: Let $\mathbf{y} \sim N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$, and let partition $\mathbf{y}$, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$ conformably as

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \text{ and } \ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

Then $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.

Proof of Theorem 1: Since $\mathbf{y} \sim N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$, its mgf is $\exp(\boldsymbol{t}'\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$. If we partition $\boldsymbol{t}$ conformably with $\mathbf{y}$, then the exponent is:

$$\boldsymbol{t}_1'\boldsymbol{\mu}_1 + \boldsymbol{t}_2'\boldsymbol{\mu}_2 + \frac{1}{2}\boldsymbol{t}_1'\boldsymbol{\Sigma}_{11}\boldsymbol{t}_1 + \frac{1}{2}\boldsymbol{t}_2'\boldsymbol{\Sigma}_{22}\boldsymbol{t}_2 + \boldsymbol{t}_1'\boldsymbol{\Sigma}_{12}\boldsymbol{t}_2.$$

If $\boldsymbol{\Sigma}_{12} = 0$ then the mgf is factorizable into separate terms involving $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, implying that $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent. Conversely, if $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent then

$$M(\mathbf{t}_1, \mathbf{0})M(\mathbf{0}, \mathbf{t}_2) = M(\mathbf{t}_1, \mathbf{t}_2).$$

Comparing this last expression to the general form above, this implies that $\boldsymbol{t}_1'\boldsymbol{\Sigma}_{12}\boldsymbol{t}_2 = 0$ for all $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, which implies that $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.

To prove **Result 4** now all we have to do is use **Result 3** with $\mathbf{C} = \binom{\mathbf{A}}{\mathbf{B}}$ and apply **Theorem 1**.

Similar results can be shown between quadratic forms based on normally distributed random variables and between linear functions and quadratic forms:

Result 5: Suppose $\mathbf{y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and let $\mathbf{q}_1 = \mathbf{y}'\mathbf{A}_1\mathbf{y}$ and $\mathbf{q}_2 = \mathbf{y}'\mathbf{A}_2\mathbf{y}$. Then $\mathbf{q}_1$ and $\mathbf{q}_2$ are independent if and only if $\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2 = \mathbf{0}$.

Result 6: Suppose $\mathbf{y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and let $\mathbf{q} = \mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{v} = \mathbf{B}\mathbf{y}$. Then $\mathbf{q}$ and $\mathbf{v}$ are independent if and only if $\mathbf{B}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{0}$.

References

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