This assignment requires Matlab. Please look at a sample solution in the next page.

1. Solve the initial value problem \( \frac{dy}{dx} = y \sin x, \ y(0) = 1, \ 0 \leq x \leq 4\pi \). Then use Euler’s method with \( N = 50 \) (i.e. \( h = 4\pi/50 \)) to solve it numerically. Compare the exact value \( y(4\pi) \) with your approximated value (which should be the last value of your sequence \( y_1, y_2, \ldots \)). Then, plot the graph of the approximated solution and the graph of the exact solution in one figure.

2. Repeat Problem 1 with the Improved Euler method.

3. Repeat Problem 1 with the Runge-Kutta method.

4. Use Euler’s method to solve \( \frac{dy}{dx} = y \cos x + x, \ y(0) = 1, \ 0 \leq x \leq 4\pi \) with \( N = 50, 500, 5000 \). Compare the three approximated values for \( y(4\pi) \), and plot the graph of the three approximated solutions in one figure.

5. Solve the Problem 4 using Euler’s method with \( N = 5000 \). (Yes, you did it in Problem 4.) Then, use the Improved Euler method with \( N = 50 \) to solve the problem. Compare the two approximated values for \( y(4\pi) \), and plot the graph of the two approximated solutions in one figure.


7. Solve \( \frac{dy}{dx} = x^2 + y^2, \ y(0) = 1, \ 0 \leq x \leq 0.5 \) using the Runge-Kutta method with \( N = 50, 200, 5000 \). Compare the three approximated values for \( y(0.5) \).

8. Repeat Problem 7 for \( 0 \leq x \leq 0.97 \). If you see anything unexpected, please read Example 5 on page 111.
Example. Solve the initial value problem \( \frac{dy}{dx} = y \cos x \), \( y(0) = 1 \), \( 0 \leq x \leq 4\pi \). Then use Euler’s method with \( N = 200 \) (i.e. \( h = 4\pi/200 \)) to solve it numerically. Compare the exact value \( y(4\pi) \) with your approximated value (which should be the last value of your sequence \( y_1, y_2, \ldots \)). Then, plot the graph of the approximated solution and the graph of the exact solution in one figure.

Here is the code I use to do the problem

```matlab
clc; clf; clear all
f=@(x,y) y*cos(x);
x_0=0;
y_0=1;
b=4*pi;

% Euler’s method
N=100;
[x,y]=Euler(f,x_0,y_0,b,N);
plot(x,y,'b','LineWidth',2,'DisplayName','Euler, N=100');
hold on

% Exact solution
x=linspace(0,4*pi,1000);
z=exp(sin(x));
plot(x,z,'r','LineWidth',2,'DisplayName','Exact');
title('Using the Euler method to \( \frac{dy}{dx}=ycos(x) \)')
xlabel('x')
ylabel('y')
axis tight
legend('show')
hold off

fprintf('Problem 1: \( \frac{dy}{dx}=ycos(x) \), \( y(0)=1 \). Euler 
')
fprintf('Approximated value = %0.6f
',y(N))
fprintf('Exact value = %0.6f
',z(end))
```

In this code, I called a function “Euler” which is defined below, and saved as in a separate file “Euler.m”.

```matlab
function [x,y]=Euler(f,x_0,y_0,b,N);
h=(b-x_0)/(N-1);
x=linspace(x_0,b,N);
y(1)=y_0;
for n=1:N-1
    y(n+1)=y(n)+h*f(x(n),y(n));
end;
end
```

What you need to turn in is the output and the graph:
Problem 1: \( \frac{dy}{dx} = y \cos(x) \), \( y(0) = 1 \). Euler

Approximated value = 0.669514

Exact value = 1.000000

**Example 2.** Use Euler’s method to solve \( \frac{dy}{dx} = y \cos(x) + x \), \( y(0) = 1 \), \( 0 \leq x \leq 4\pi \) with \( N = 50, 500, 5000 \). Compare the three approximated values for \( y(4\pi) \), and plot the graph of the three approximated solutions in one figure.

Here is the code:

```matlab
clc; clear all; clf
f=@(x,y) y*sin(x)+x;
x_0=0;
y_0=1;
b=4*pi;
N=50;
[x,y]=Euler(f,x_0,y_0,b,N);
fprintf('Problem 4: \( \frac{dy}{dx} = y \sin(x) + x \), \( y(0) = 1 \). Euler \n')
fprintf(' N \ y(4\pi) \n')
fprintf('%5d %0.6f
', N,y(N));
plot(x,y,'b','LineWidth',2,'DisplayName','N=100');
hold on
N=500;
[x,y]=Euler(f,x_0,y_0,b,N);
fprintf('%5d %0.6f
', N,y(N));
plot(x,y,'b','LineWidth',2,'DisplayName','N=500');
N=5000;
[x,y]=Euler(f,x_0,y_0,b,N);
fprintf('%5d %0.6f
', N,y(N));
plot(x,y,'b','LineWidth',2,'DisplayName','N=5000');
hold on
```

```matlab
```
plot(x,y,'r','LineWidth',2,'DisplayName','N=200');

N=5000;
[x,y]=Euler(f,x_0,y_0,b,N);
fprintf('%5d %0.6f
', N,y(N));
plot(x,y,'k','LineWidth',2,'DisplayName','N=5000');

title('Using the Euler method to dy/dx=ysin(x)+x')
xlabel('x')
ylabel('y')
axis tight
legend('show')
hold off

Things to turn in: the output and the graph:

Problem 4: dy/dx=ysin(x)+x, y(0)=1. Euler

<table>
<thead>
<tr>
<th>N</th>
<th>y(4pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>29.181331</td>
</tr>
<tr>
<td>500</td>
<td>36.765816</td>
</tr>
<tr>
<td>5000</td>
<td>37.672109</td>
</tr>
</tbody>
</table>

Using the Euler method to dy/dx=ysin(x)+x

![Graph Image]
Note that for Euler’s method, you may use the function \textit{Euler} I defined. However, in order to solve problems for Improved Euler or Runge-Kutta, you need to define the functions by yourself, say \textit{ImprovedEuler} and \textit{RungeKutta}, and call these functions when needed.