Lecture 3  Separable equations  1/17/2018

In general a first order differential equation
\[ \frac{dy}{dx} = f(x, y) \]

If \( f(x, y) \) can be written as \( g(x) \cdot h(y) \), then
\[ \frac{dy}{dx} = g(x) \cdot h(y) \]
is called separable eq.

E.g.
1) \[ \frac{dy}{dx} = \frac{4-2x}{3y^2-5} \]
2) \[ \frac{dy}{dx} = 6e^{2x-y} \quad (6e^{2x-y} = 6e^{2x} \cdot e^{-y}) \]
3) \[ \frac{dy}{dx} = x+y+xy \quad (1+x+y+xy = 1+x+y(1+x) = (1+x)(1+y)) \]
However \( \frac{dy}{dx} = x+y \) is not separable.

Ex 1. Solve \( \frac{dy}{dx} = \frac{4-2x}{3y^2-5} \)

By multiplying \( 3y^2-5 \), we have
\[ \int (3y^2-5) \, dy = \int (4-2x) \, dx \]
\[ y^3-5y = 4x-x^2 + C \]
\[ y^3-5y - 4x + x^2 = C \]

Leave your answer like this, if you cannot solve for \( y \) explicitly.
Ex 2. Solve initial value problem
\[
\frac{dy}{dx} = 6e^{2x-y}, \quad y(0)=1
\]
\[\frac{dy}{dx} = 6e^{2x} \cdot e^{-y}. \text{ By multiplying } e^y, \text{ we have} \]
\[\int e^y \, dy = \int 6e^{2x} \, dx \]
\[e^y = 3e^{2x} + C. \text{ Since } y(0) = 1, \quad e^1 = 3e^0 + C \]
\[C = e - 3 \]
\[y = \ln(3e^{2x} + C) = \ln(3e^{2x} + e - 3) \]

Ex 3. Find all the solutions to
\[
\frac{dy}{dx} = 1 + x + y + xy
\]
\[\frac{dy}{dx} = (1+x)(1+y) \]

1. If \(1+y=0\), \(y=-1\). Right-hand side = 0, left-hand side = 0. So \(y=-1\) is a solution.

2. If \(1+y \neq 0\), by dividing both sides by \(1+y\), we get
\[
\int \frac{dy}{1+y} = \int (1+x) \, dx
\]
\[\ln|1+y| = x + \frac{x^2}{2} + C \]
\[|1+y| = e^{x + \frac{x^2}{2} + C} = e^{x + \frac{x^2}{2}} \cdot e^C = C_1 \cdot e^{x + \frac{x^2}{2}} \]
where \(C_1 = e^C\) is a positive constant.
\[1+y = \pm C_2 \cdot e^{x + \frac{x^2}{2}} = C_2 \cdot e^{x + \frac{x^2}{2}} \]
where \(C_2\) is a constant, \(C_2 \neq 0\).
Thus \[ y = -1 + c_2 e^{x + \frac{x^2}{3}} \] for \( c_2 \neq 0 \).

In conclusion, the equation \( \frac{dy}{dx} = 1 + x + y + xy \) has solutions:

1. \( y = -1 \)
2. \( y = -1 + c_2 e^{x + \frac{x^2}{3}} \), \( c_2 \neq 0 \)

In this particular case, (1) and (2) can be combined into

\[ y = -1 + c_3 e^{x + \frac{x^2}{3}} \], \( c_3 \) is any real number

Ex 4. Find all the solutions to

\[ \frac{dy}{dx} = 6x(y-1)^{2/3} \]

1. \( y = 1 \) is a solution

2. If \( y \neq 1 \),

\[ \int \frac{1}{(y-1)^{2/3}} \, dy = \int 6x \, dx \]

\[ \frac{\left( y-1 \right)^{3/2}}{-3/2 + 1} = 3x^2 + C \]

i.e.

\[ 3(y-1)^{1/3} = 3x^2 + C \]

\[ (y-1)^{1/3} = x^2 + \frac{C}{3} \]

\[ y-1 = (x^2 + \frac{C}{3})^3 \]

\[ y = 1 + (x^2 + \frac{C}{3})^3, \text{ real} \]
Ex 5. An application of separable equations to Newton's law of cooling/ heating.

Newton's law of cooling says

\[ \frac{dT}{dt} = k(A-T) \]

where \( T \) is the temperature of the coffee in the cup, \( A \) is the room temperature, \( k \) is a constant. Let's solve this problem.

1. If \( A-T=0 \), then \( T=A \) is a solution.
2. If \( A-T\neq0 \), then

\[ \int \frac{dT}{A-T} = \int k \, dt \]

\[ -\ln|A-T| = kt + C \]

\[ \ln|A-T| = \ln|T-A| = -kt - C \]

\[ |T-A| = e^{-kt-C} = e^{-kt}e^{-C} \]

\[ T-A = \pm e^{-C}e^{-kt} = c_1e^{-kt} \]

Thus, \[ T = A + c_1 e^{-kt} \]

where \( c_1 = \pm e^{-C} \neq 0 \)