Lecture 7. Exact Equations 1/26/2018

So far, we have studied the methods of solving equations of the form
1. \( \frac{dy}{dx} = g(x)h(y) \) by separating variables
2. \( \frac{dy}{dx} + p(x)y = q(x) \) by using integration factor \( u(x) = e^{\int p(x)dx} \)
3. \( \frac{dy}{dx} = f\left(\frac{y}{x}\right) \) by substitution \( v = \frac{y}{x} \), so \( \frac{dy}{dx} = v + x\frac{dv}{dx} \)
4. \( \frac{dy}{dx} = f(ax+by+c) \) by substitution \( v = ax+by+c \)

There are more substitution techniques in §1.6, such as Bernoulli equations. Please read §1.6. Here I introduce one more example.

Ex. 1 Solve \( y'' = (x+y')^2 \)

Let \( v = y' \), then \( y'' = v' = \frac{dv}{dx} \). Plug in, we get

\[
\frac{dv}{dx} = (x+v)^2
\]

Let \( u = x+v \). \( \frac{du}{dx} = 1 + \frac{dv}{dx} \), \( \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx} = \frac{dv}{dx} - 1 \). Thus

\[
\frac{dv}{dx} - 1 = u^2 \Rightarrow \frac{dv}{dx} = 1 + u^2
\]

\[
\int \frac{1}{1+u^2} \, du = \int dx \quad \tan^{-1} u = x + C
\]

\( u = \tan(x+C) \) i.e. \( x + v = \tan(x+C) \)

\( v = \tan(x+C) - x \), i.e. \( y' = \tan(x+C) - x \)

\( y = \int (\tan(x+C) - x) \, dx \)

\( = \int \frac{\sin(x+C)}{\cos(x+C)} \, dx - \frac{1}{2} x^2 \), let \( \cos(x+C) = w \)

\( = \int \frac{1}{w} \, dw - \frac{1}{2} x^2 \)

\( = -\ln|w| - \frac{1}{2} x^2 + C_1 = -\ln|\cos(x+C)| - \frac{1}{2} x^2 + C_1 \)
If \( F(x,y) \) is a function of two variables, e.g.
\[
F(x,y) = x^2 + y^2 - 2xy + 1
\]
We can view \( F \) as a function of \( x \) (with \( y \) fixed) and consider the derivative with respect to \( x \). We call such a derivative as "partial derivative with respect to \( x \)" and denote by \( \frac{\partial F}{\partial x} \). For \( F \) given above
\[
\frac{\partial F}{\partial x} = 2x + 0 - 2y + 0 = 2x - 2y
\]
We can define \( \frac{\partial F}{\partial y} \) in a similar way.
\[
\frac{\partial F}{\partial y} = 0 + 2y - 2x + 0 = 2y - 2x
\]

Ex. 2 \( G(x,y) = xy + e^x + \sin(x+2y) - 3 \).
Find
\[
\frac{\partial G}{\partial x} = y + e^x + \cos(x+2y) \cdot 1 - 0
\]
\[
\frac{\partial G}{\partial y} = x + 0 + \cos(x+2y) \cdot 2 - 0
\]

One application of partial derivatives is that we can find \( \frac{dy}{dx} \) for function \( y \) that is determined by \( F(x,y) = c \) without solving \( y \) explicitly

\[
\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}
\]

Ex. 3. If \( x^2 + y^3 - 5y - 1 = 0 \). Find \( \frac{dy}{dx} \)
\[
\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x}{3y^2-5}
\]

Ex4. Solve the equation

\[2x + (3y^2-5)y' = 0\]

By Ex3. the solution is \[x^2 + y^3 - 5y - 1 = 0\]

Ex5. Solve the equation

\[\frac{f(x)}{3x^2y^3 + y^4} + \frac{f(x)}{3x^3y^2 + y^4 + 4xy^3} y' = 0\]

In general \[M(x,y) + N(x,y) y' = 0\] (*)

If \(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\), then the equation (*) is called an exact equation and we can find a function \(F(x,y)\) so that \(F(x,y) = c\) is the solution to (*)

Back to Ex5.

\[M(x,y) = 3x^2y^3 + y^4, \quad \frac{\partial M}{\partial y} = 9x^2y^2 + 4y^3\]
\[N(x,y) = 3x^3y^2 + y^4 + 4xy^3, \quad \frac{\partial N}{\partial x} = 9x^2y^2 + 0 + 4y^3\]

\[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\] so the equation in Ex5 is exact.

\[F(x,y) = \int (3x^2y^3 + y^4)\,dx = x^3y^3 + y^4x + h(y)\]
\[= \int (3x^3y^2 + y^4 + 4xy^3)\,dy = ax^3y^3 + y^4 + k(x)\]
\[= x^3y^3 + \frac{1}{5}y^5 + xy^4 + F(x)\]
So \( x^3y^3 + y^4x + h(y) = x^3y^3 + \frac{1}{5}y^5 + xy^4 + k(x) \)

Thus \( h(y) = \frac{1}{5}y^5 \). \( k(x) = 0 \)

Therefore \( F(x, y) = x^3y^3 + y^4x + \frac{1}{5}y^5 \)

and the solution is

\[
\boxed{x^3y^3 + y^4x + \frac{1}{5}y^5 = C}
\]