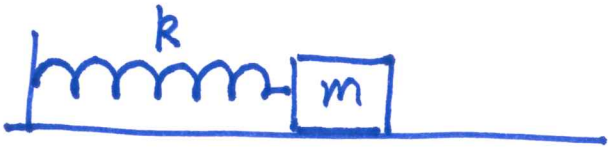
§7.6. Delta function $\delta_a(t)$. Wed 11/30



$m x'' + kx = 0 \quad x'(0) = x_1, \; x(0) = x_0$

if there is no external force.

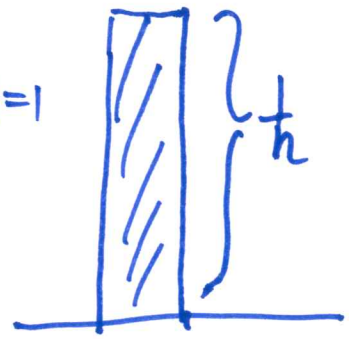
If there is external force $f(t)$

$$m x'' + kx = f(t) \qquad x'(0) = x_1, \; x(0) = x_0$$

If

$$f(t) = \begin{cases} \frac{1}{h} & a \le t \le a+h \\ 0 & \text{o.w.} \end{cases}$$

$\int_{-\infty}^{\infty} f(t)\,dt = 1$



Let $h \to 0$. We say $\delta_a(t)$ is a "limit" of these $f(t)$ as $h \to 0$.

$\delta_a(t)$ is not a function in the usual sense but it has the property.

① $\int_{-\infty}^{\infty} \delta_a(t)\,dt = 1$

② $\delta_a(t) = 0$ if $t \neq a$

So. $$\delta_a(t) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}$$

If $a=0$, we simply denote it by $\delta(t)$

$$\int_{-\infty}^{\infty} 3\,\delta_a(t)\,dt = 3\int_{-\infty}^{\infty}\delta_a(t)\,dt = 3$$

$$3\cdot\delta_a(t) = 3\cdot\begin{cases}\infty & t=a\\ 0 & t\neq a\end{cases}$$

$$= \begin{cases}3\cdot\infty & t=a\\ 0 & t\neq a\end{cases}$$

$$\neq \begin{cases}\infty & t=a\\ 0 & t\neq a\end{cases} = \delta_a(t)$$

The most useful fact:

$$f(t)\,\delta_a(t) \equiv f(a)\,\delta_a(t)$$

(for $t=a$, of course they are equal
for $t\neq a$, both side $=0$.)

Ex 1. Solve $x'' + 4x = \delta(t)$. $x(0)=0$. $x'(0)=0$

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\delta(t)\}$$

$$(s^2X - 0\cdot s - 0) + 4X = 1$$

$$(s^2+4)X = 1. \quad X = \frac{1}{s^2+4}. \quad x(t) = \frac{1}{2}\sin(2t)$$

$$\mathcal{L}\{\delta_a(t)\} = \int_0^\infty \boxed{e^{-st}\delta_a(t)}\,dt$$

$$= \int_0^\infty \boxed{e^{-sa}\delta_a(t)}\,dt$$

$$= e^{-sa}\int_0^\infty \delta_a(t)\,dt$$

$$= e^{-sa}\int_{-\infty}^\infty \delta_a(t)\,dt = e^{-sa}$$

$$\mathcal{L}\{\delta_a(t)\} = e^{-sa}. \qquad \boxed{\mathcal{L}\{\delta(t)\} = 1}$$

This should be compared with

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{?\} = s$$

Ex 2 Solve $x'' + 4x = \delta(t) + \delta(t-\pi)$

$x(0) = 0, \quad x'(0) = 0$

$\delta(t-\pi) = \delta_\pi(t)$

$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\delta(t)\} + \mathcal{L}\{\delta_\pi(t)\}$

$s^2X + 4X = 1 + e^{-\pi s}$

$X = \frac{1}{s^2+4} + \frac{1}{s^2+4} \cdot e^{-\pi s}$

$x(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}\sin 2(t-\pi) \cdot u_\pi(t)$

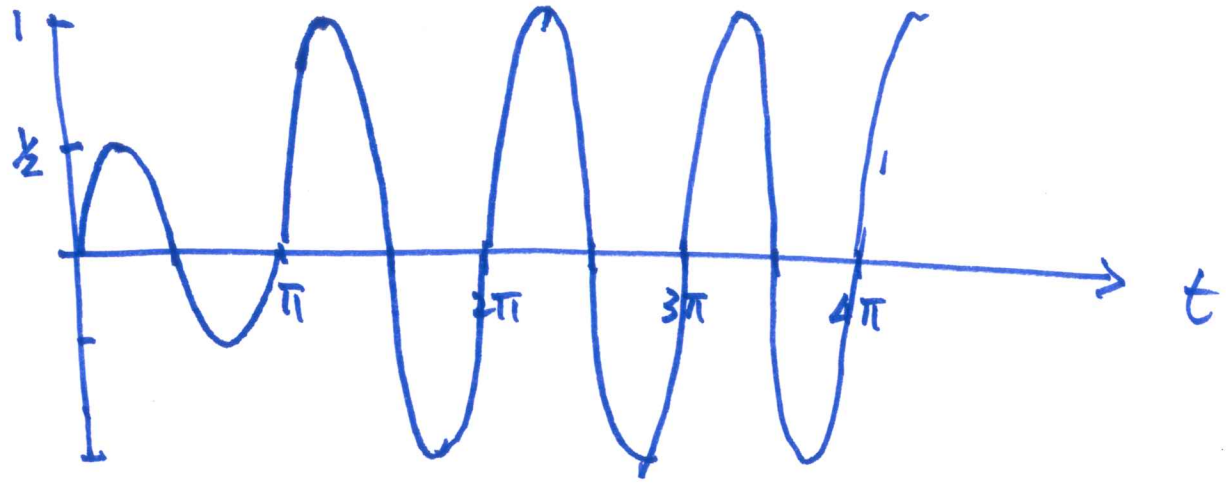
$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}F(s)$

$\mathcal{L}^{-1}\{F(s)\,e^{-as}\} = f(t-a)\,u_a(t)$

$x(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}\sin(2t) \cdot u_\pi(t)$

$$= \begin{cases} \frac{1}{2}\sin(2t) & 0 \le t < \pi \\ \sin(2t) & t > \pi \end{cases}$$

$\boxed{\sin(2t-2\pi) = \sin(2t)}$


1
½
π
2π
3π
4π
t

Ex3. Solve $x''+2x'+x = t+\delta(t)$, $x(0)=0$, $x'(0)=1$

$$(s^2X-1)+2(sX-0)+X = \frac{1}{s^2}+1$$

$$(s^2+2s+1)X = \frac{1}{s^2}+2$$

$$X = \frac{1}{s^2(s^2+2s+1)} + \frac{2}{s^2+2s+1}$$

$$= \frac{1}{s^2(s+1)^2} + \frac{2}{(s+1)^2}.$$

$$\boxed{= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}}$$

$$\frac{1}{s^2(s+1)^2} = \left[\frac{1}{s(s+1)}\right]^2 = \left(\frac{1}{s} - \frac{1}{s+1}\right)^2$$

$$= \frac{1}{s^2} + \frac{1}{(s+1)^2} - 2\cdot\frac{1}{s(s+1)}$$

$$= \frac{1}{s^2} + \frac{1}{(s+1)^2} - 2\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= \frac{1}{s^2} + \frac{1}{(s+1)^2} - \frac{2}{s} + \frac{2}{s+1}$$

Thus $$X = \frac{1}{s^2} + \frac{3}{(s+1)^2} - \frac{2}{s} + \frac{2}{s+1}$$

$$x(t) = t + 3te^{-t} - 2 + 2e^{-t}$$