§7.1 Laplace transforms

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ provided that the improper integral converges.

① $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ $\quad n = 0, 1, 2, \ldots \quad 0! = 1, \; 1! = 1.$

② $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ $\quad (s > a)$

Gamma function $\Gamma(x)$ for $x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \qquad (\text{converges for } x > 0)$$

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\,dt \\
&= t^x(-e^{-t})\Big|_0^\infty - \int_0^\infty x t^{x-1}(-e^{-t})\,dt \\
&= \lim_{M\to\infty}\left[t^x(-e^{-t})\Big|_{t=0}^{M}\right] + x\int_0^\infty t^{x-1}e^{-t}\,dt \\
&= 0 + x\Gamma(x)
\end{aligned}$$

$\Gamma(x+1) = x\Gamma(x) \qquad x > 0$

If $x = n$ integer

$$\begin{aligned}
\Gamma(n+1) &= n\underline{\Gamma(n)} = n\underline{(n-1)\Gamma(n-1)} = \cdots \\
&= n(n-1)(n-2)\cdots 1\,\Gamma(1) \\
&= n!
\end{aligned}$$

$\Gamma(1) = 1$

$$\boxed{\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(n+1) = n!}$$

Ex 1. Find the Laplace transform of $f(t) = t^a$, $a>0$
not necessary integer. e.g $t^{\sqrt{2}}$

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a dt$$

Let $y = st$, $t = \frac{1}{s} y$

$dt = \frac{1}{s} dy$

for $s>0$

$$= \int_0^\infty e^{-y} \left(\frac{y}{s}\right)^a \frac{1}{s} dy$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-y} y^a dy$$

$$= \frac{1}{s^{a+1}} \cdot \Gamma(a+1)$$

$$\boxed{\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}} \quad (s>0)} \quad a>-1$$

e.g

$$\mathcal{L}\{t^{\sqrt{2}}\} = \frac{\Gamma(\sqrt{2}+1)}{s^{\sqrt{2}+1}} = \frac{\sqrt{2}\,\Gamma(\sqrt{2})}{s^{\sqrt{2}+1}}.$$

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\Gamma(\frac{1}{2})}{\sqrt{s}}$$

$$\mathcal{L}\{\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s\sqrt{s}} = \frac{\Gamma(\frac{1}{2})}{2s\sqrt{s}}$$

How about

$\mathcal{L}\{\cos bt\}$. $\mathcal{L}\{\sin bt\}$.

$\mathcal{L}\{e^{at}\cos bt\}$. $\mathcal{L}\{e^{at}\sin bt\}$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s > a$$

$$\mathcal{L}\{e^{bit}\} = \frac{1}{s-bi}$$

By Euler's identity

$$e^{bit} = \cos bt + i \sin bt$$

$$\mathcal{L}\{\cos bt + i \sin bt\} = \frac{1}{s-bi} = \frac{s+bi}{(s-bi)(s+bi)}$$

$$\mathcal{L}\{\cos bt\} + i\mathcal{L}\{\sin bt\} = \frac{s+bi}{s^2+b^2} = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}$$

Matching the real parts and the imaginary parts.

$$\boxed{\begin{aligned} \mathcal{L}\{\cos bt\} &= \frac{s}{s^2+b^2} \\ \mathcal{L}\{\cancel{\cos}\sin bt\} &= \frac{b}{s^2+b^2} \end{aligned}}$$

Note that

$$\mathcal{L}\{\cos bt\} = \int_0^\infty e^{-st} \cos bt \, dt$$

$$= \frac{s}{s^2+b^2}$$

(we could obtain this directly by integration by parts twice)

$$\mathcal{L}\{\sin bt\} = \int_0^\infty e^{-st} \sin bt \, dt$$

$$= \frac{b}{s^2+b^2}.$$

$$\mathcal{L}\{e^{at}\cos bt\} = \int_0^\infty e^{-st}\cdot e^{at}\cos bt\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\cos bt\,dt$$

$$= \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\sin bt\} = \int_0^\infty e^{-st}e^{at}\sin bt\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\sin bt\,dt$$

$$= \frac{b}{(s-a)^2+b^2}.$$

$$\mathcal{L}\{e^{at}\cdot t^n\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

<u>Example 2</u> Find the Laplace transform of

$$f(t) = 3e^{2t} + 2\sin^2 3t$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3e^{2t} + 2\sin^2 3t\}$$

$$= 3\mathcal{L}\{e^{2t}\} + 2\mathcal{L}\{\sin^2 3t\}$$

$$= 3\cdot\frac{1}{s-2} + \cancel{2}\mathcal{L}\left\{\frac{1-\cos 6t}{\cancel{2}}\right\}$$

$$= \frac{3}{s-2} + \mathcal{L}\{1\} - \mathcal{L}\{\cos 6t\} = \frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2+6^2}$$

Recall that

$$\sin^2\beta t = \frac{1-\cos 2\beta t}{2}$$

A function f is said to have an exponential growth rate if there exist constants M and A such that

$$|f(t)| \le Me^{At} \quad \text{for all} \quad t \ge 0.$$

e.g If $f(t)$ is a polynomial of t. then

$$\lim_{t\to\infty} \frac{f(t)}{e^t} \overset{\text{L'Hospital}}{=\!=\!=} 0. \quad \text{So} \quad \left|\frac{f(t)}{e^t}\right| \le M.$$

$$\Rightarrow |f(t)| \le Me^t.$$

e.g. $e^{\frac{t^2+1}{t+1}}$ has an exponential growth rate.

$$\frac{t^2+1}{t+1} \le \frac{t^2+2t+1}{t+1} = t+1 \quad t \ge 0.$$

$$e^{\frac{t^2+1}{t+1}} \le e^{t+1} = e^1 \cdot e^t = Me^t$$

However e^{t^2} ~~has not~~ is not of ~~expon.~~ exponential growth rate.

Thm. If $f(t)$ is continuous and has exponential growth rate. then $\mathcal{L}\{f(t)\}$ exists and is unique.