7.1 Laplace transforms

\[ L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt \quad \text{provided that the improper integral converges.} \]

0. \( L\{t^n\} = \frac{n!}{s^{n+1}} \quad n=0, 1, 2, \ldots \quad 0! = 1, \ 1! = 1 \)

1. \( L\{e^{at}\} = \frac{1}{s-a} \quad (s > a) \)

Gamma function \( \Gamma(x) \) for \( x > 0 \)

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad \text{(converges for } x > 0) \]

\[ \Gamma(x+1) = \int_0^\infty t^x e^{-t} \, dt \]

\[ = [t^x e^{-t}]_0^\infty - \int_0^\infty x t^{x-1} (-e^{-t}) \, dt \]

\[ = \lim_{M \to \infty} \left[ t^x (-e^{-t}) \bigg|_0^M \right] + x \int_0^\infty t^{x-1} e^{-t} \, dt \]

\[ = 0 + x \Gamma(x) \]

\[ \Gamma(x+1) = x \Gamma(x) \quad x > 0 \]

If \( x = n \) integer

\[ \Gamma(n+1) = n \Gamma(n) = n (n-1) \Gamma(n-1) = \ldots \]

\[ = n (n-1) (n-2) \ldots 1 \Gamma(1) \]

\[ = n! \]

\[ \Gamma(1) = 1 \]

\[ \boxed{\Gamma(x+1) = x \Gamma(x), \quad \Gamma(n+1) = n!} \]
Ex 1. Find the Laplace transform of \( f(t) = t^a \), \( a > 0 \) not necessarily integer. e.g. \( t^{\sqrt{2}} \)

\[
\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a \, dt
\]

Let \( y = st \), \( t = \frac{1}{s} y \)

\[
dt = \frac{1}{s} dy \text{ for } s > 0
\]

\[
= \frac{1}{s^{a+1}} \int_0^\infty e^{-y} y^a \, dy
\]

\[
= \frac{\Gamma(a+1)}{s^{a+1}} \quad (s > 0)
\]

a \(-1
\]

e.g.
\[
\mathcal{L}\{t^{\sqrt{2}}\} = \frac{\Gamma(\sqrt{2}+1)}{s^{\sqrt{2}+1}} = \frac{\sqrt{2} \Gamma(\sqrt{2})}{s^{\sqrt{2}+1}}.
\]

\[
\mathcal{L}\{t^{\frac{1}{3}}\} = \frac{\Gamma(\frac{1}{3})}{s^{\frac{1}{3}}}
\]

\[
\mathcal{L}\{t^{\frac{3}{2}}\} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{1}{2} \frac{\Gamma(\frac{1}{2})}{s^{3/2}} = \frac{\Gamma(\frac{1}{2})}{2s^{3/2}}
\]

How about

\[
\mathcal{L}\{\cos bt\}, \mathcal{L}\{\sin bt\}, \mathcal{L}\{e^{at}\} (s > bt), \mathcal{L}\{e^{at} \sin bt\}
\]
\[ L\{e^{at}\} = \frac{1}{s-a}, \quad s > a \]

\[ L\{e^{bit}\} = \frac{1}{s-bi} \]

By Euler's identity,
\[ e^{bit} = \cos bt + i\sin bt \]

\[ L\{\cos bt + i\sin bt\} = \frac{1}{s-bi} = \frac{s+bi}{(s-bi)(s+bi)} \]

\[ L\{\cos bt\} + i L\{\sin bt\} = \frac{s+bi}{s^2+b^2} = \frac{s}{s^2+b^2} + i \frac{b}{s^2+b^2} \]

Matching the real parts and the imaginary parts,

\[ L\{\cos bt\} = \frac{s}{s^2+b^2} \]

\[ L\{\sin bt\} = \frac{b}{s^2+b^2} \]

Note that,
\[ L\{\cos bt\} = \int_{0}^{\infty} e^{-st} \cos bt \, dt \]

\[ = \frac{s}{s^2+b^2} \] (We could obtain this directly by integration by parts twice.

\[ L\{\sin bt\} = \int_{0}^{\infty} e^{-st} \sin bt \, dt \]

\[ = \frac{b}{s^2+b^2}. \]
\[ L \{ e^{at} \cos bt \} = \int_0^\infty e^{-st} e^{at} \cos bt \, dt \]
\[ = \int_0^\infty e^{-st} \cos bt \, dt \]
\[ = \frac{s-a}{(s-a)^2 + b^2} \]

\[ L \{ e^{at} \sin bt \} = \int_0^\infty e^{-st} e^{at} \sin bt \, dt \]
\[ = \int_0^\infty e^{-(s-a)t} \sin bt \, dt \]
\[ = \frac{b}{(s-a)^2 + b^2} \]

\[ L \{ e^{at} \cdot t^n \} = \frac{n!}{(s-a)^{n+1}} \quad \text{for } L \{ e^{at} \} = \frac{n!}{s^{n+1}} \]

**Example 2** Find the Laplace transform of

\[ f(t) = 3e^{2t} + 2\sin^2 3t \]

\[ L\{ f(t)^2 \} = L\{ 3e^{2t} + 2\sin^2 3t \} \]
\[ = 3 L\{ e^{2t} \} + 2 L\{ \sin^2 3t \} \]
\[ = 3 \cdot \frac{1}{s-2} + 2 \cdot L\{ \frac{1 - \cos 6t}{2} \} \]

Recall that \[ \frac{s^2 + 1}{s^2 + 6^2} - L\{ \cos 6t \} = \frac{3}{s^2 + 1} - \frac{s}{s^2 + 6^2} \]

\[ \sin^2 \beta t = \frac{1 - \cos 2\beta t}{2} \]
A function is said to have an exponential growth rate if there exist constants \( M \) and \( A \) such that

\[ |f(t)| \leq Me^{At} \quad \text{for all } t \geq 0. \]

E.g. If \( f(t) \) is a polynomial of \( t \), then

\[
\lim_{t \to 0} \frac{f(t)}{e^t} = 0 \quad \text{so} \quad \left| \frac{f(t)}{e^t} \right| \leq M.
\]

\[ \Rightarrow \left| f(t) \right| \leq Me^t. \]

E.g. \( e^{\frac{t^2}{t+1}} \) has an exponential growth rate.

\[
\frac{t^2+1}{t+1} \leq \frac{t^2+2t+1}{t+1} = t+1, \quad t \geq 0.
\]

\[ e^{\frac{t^2}{t+1}} \leq e^{t+1} = e^t \cdot e^t = Me^t. \]

However, \( e^{t^2} \) is not. is not of exponential growth rate.

Thus, if \( f(t) \) is continuous and has exponential growthrate, then \( \lim_{t \to 0} f(t) \) exists and is unique.