1. Let $N(t)$ be a Poisson process with mean $\lambda$. Calculate $\text{Var}[3N(t) + N(2t)]$

Solution:
\[
\text{Var}[3N(t) + N(2t)] = \text{Var}[4N(t) + N(2t) - N(t)] = \text{Var}[4N(t)] + \text{Var}[N(2t) - N(t)] = 16\lambda t + \text{Var}[N(t)] = 17\lambda t.
\]

2. Let $W(t)$ be a compound process defined as
\[
W(t) = \sum_{i=1}^{N(t)} X_i
\]
where $N(t)$ is a Poisson process with rate $\lambda = 4$, and $X_i$ are i.i.d. uniformly distributed on $[1, 2]$. Find $\mathbb{E}[W(3)^2]$.

Solution:
\[
\mathbb{E}W(3) = \lambda(3)\mathbb{E}X_1 = 12 \cdot \frac{3}{2} = 18.
\]
\[
\text{Var}[W(3)] = \lambda(3)\mathbb{E}X_1^2 = 12 \cdot \int_1^2 s^2 \, ds = 28.
\]
Thus, $\mathbb{E}[W(3)^2] = 18^2 + 28 = 352$.

3. Let $T_1$ be the first arrival time of a non-homogeneous Poisson process $N(t)$ with intensity function $\lambda(t) = t$. Find $\mathbb{E}(T_1^2)$. (Hint: Find the distribution of $T_1$.)

Solution: $\mathbb{P}(T_1 > t) = \mathbb{P}(N(t) = 0) = e^{-\int_0^t \lambda(s) \, ds} = t^2/2$. Thus, $f_{T_1}(t) = te^{-t^2/2}$. Therefore
\[
\mathbb{E}(T_1^2) = \int_0^\infty t^2 \cdot te^{-t^2/2} \, dt = \int_0^\infty 2xe^{-x} \, dx = 2.
\]

4. Bus Route A arrives a certain bus stop according to a Poisson process $N_1(t)$ with rate $\lambda_1$. Independently, Bus Route B arrives the stop according to a Poisson process $N_2(t)$ with rate $\lambda_2$. If a person takes Bus A, it takes time $s_1$ to get home; if he takes Bus B, it take $T$ to get home, where $T$ is uniformly distributed on $[s_2, s_3]$.

(a) If a person decides to take Bus A only, what is the expected time needed to get home.
Solution:

\[ E(W) = E[X_1 + \text{Ride}] = \frac{1}{\lambda_1} + s_1. \]

(b) If a person decides to take Bus B only, what is the expected time needed to get home.

Solution:

\[ E(W) = E[Y_1 + \text{Ride}] = \frac{1}{\lambda_2} + \frac{s_2 + s_3}{2}. \]

(c) What is the expected waiting time for the first bus to arrive? (Hint: Consider \( N(t) = N_1(t) + N_2(t) \).)

Solution: Let \( N(t) = N_1(t) + N_2(t) \). Then \( N(t) \) is a Poisson process with rate \( \lambda_1 + \lambda_2 \). The arrive time has mean \( 1/(\lambda + \lambda_2) \).

(d) If a person decides to take whichever bus arrives first, what is the expected time needed to get home? (Hint: \( \mathbb{P}(N_1(t) = 1|N(t) = 1) = \lambda_1/(\lambda_1 + \lambda_2) \).)

Solution:

\[ E(W) = \frac{1}{\lambda_1 + \lambda_2} + E[\text{Ride}] \]
\[ = \frac{1}{\lambda_1 + \lambda_2} + E[\text{Ride}|\text{BusA}] \cdot \mathbb{P}(\text{BusA comes first}) + E[\text{Ride}|\text{BusB}] \cdot \mathbb{P}(\text{BusB comes first}) \]
\[ = \frac{1}{\lambda_1 + \lambda_2} + s_1 \cdot \mathbb{P}(N_1(t) = 1|N(t) = 1) + \frac{s_2 + s_3}{2} \cdot \mathbb{P}(N_2(t) = 1|N(t) = 1) \]
\[ = \frac{1}{\lambda_1 + \lambda_2} + s_1 \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} + \left( \frac{s_2 + s_3}{2} \right) \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2}. \]

5. True or False: A renewal process always has independent increment.

Solution: False.