Review Problems–Part II (for Friday)

1. Finding stationary distribution for finite state continuous-time Markov chain. Related material: Formulas (5.5.1), (5.5.3)

Consider a continuous-time Markov chain with state space \{1, 2, 3, 4\} and parameters

\[
(q_{ij}) = \begin{pmatrix}
0 & 1 & 1 & 0 \\
6 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 \\
4 & 5 & 1 & 0
\end{pmatrix},
\]

Find the long-run proportion of time the chain is in state 1.

**Solution 1:** \(v_1 = 2, v_2 = 10, v_3 = 1, v_4 = 10\). The corresponding \(P_{ij}\) for the discrete chain is

\[
(P_{ij}) = \begin{pmatrix}
0 & .5 & .5 & 0 \\
.6 & 0 & .4 & 0 \\
0 & 0 & 0 & 1 \\
.4 & .5 & 1 & 0
\end{pmatrix},
\]

which is a doubly stochastic matrix. Thus, \(\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1/4\). Hence by Formula (5.5.1) we have

\[P_1 = \frac{\pi_1/v_1}{\pi_1/v_1 + \pi_2/v_2 + \pi_3/v_3 + \pi_4/v_4} = \frac{5}{17}.\]

**Solution 2:** \(v_1 = 2, v_2 = 10, v_3 = 1, v_4 = 10\). Formula (5.5.3) gives us

\[
\begin{cases}
2P_1 = 6P_2 + 4P_3 \\
10P_2 = P_1 + 5P_4 \\
P_3 = P_1 + 4P_2 + P_4 \\
10P_4 = P_3
\end{cases}
\]

Solving for this system, we obtain \(P_1 = 5/17\). (\(P_2 = 1/17, P_3 = 10/17\) and \(P_4 = 1/17\).)

2. Let \(N(t)\) be a Poisson process with parameter \(\lambda\). Find \(\text{cov}(N(3), N(4))\).

3. Let \(N(t)\) be a non-homogeneous Poisson process with intensity function \(\lambda(t) = \frac{1}{1+t}\). Find \(\mathbb{P}(N(3) = 4|N(2) = 2)\).

4. The definition of renewal processes, the definition of stopping time; The proofs of Proposition 3.2.1 and Theorem 3.3.2

Prove that if \(N\) is a stopping time for i.i.d random variables \(X_1, X_2, \ldots\) then for any \(k \geq 0\) we have

\[
\mathbb{E} \sum_{i=1}^{N+k} X_i = (\mathbb{E}N + k)\mathbb{E}X_1,
\]

but it is usually not true if \(k < 0\).

5. Definitions of transient/recurrent states, positive/null recurrent states, ergodic chains, irreducible chains; accessible/communicate, etc. The calculation of \(d(i), f_{ij}^n\) for \(f_{ij}\) for finite Markov chain using a diagram.