Solution to HW 3

Problem 3.3

\[ \mathbb{P}(X_{N(t)+1} \geq x) = \sum_{n=0}^{\infty} \mathbb{P}(X_{n+1} \geq x, N(t) = n). \]

Note that if \( n \neq 0 \), then the event \( X_{n+1} \geq x \) and the event \( N(t) = n \) are independent, we have

\[ \mathbb{P}(X_{n+1} \geq x, N(t) = n) = \mathbb{P}(X_{n+1} \geq x) \cdot \mathbb{P}(N(t) = n) = \mathcal{F}(x) \mathbb{P}(N(t) = n). \]

However, if \( n = 0 \), then the event \( X_{n+1} = 0 \) and the event \( N(t) = 0 \) are not independent. In this case we have

\[ \mathbb{P}(X_{1} \geq x; N(t) = 0) = \mathbb{P}(X_{1} \geq x; X_{1} > t). \]

If \( x \geq t \), then \( \mathbb{P}(X_{1} \geq x; N(t) = 0) = \mathcal{F}(x) \); if \( x < t \), then \( \mathbb{P}(X_{1} \geq x; N(t) = 0) = \mathbb{P}(N(t) = 0) \).

In any case we have \( \mathbb{P}(X_{1} \geq x; N(t) = 0) \geq \mathcal{F}(x) \mathbb{P}(N(t) = 0) \). Hence

\[ \mathbb{P}(X_{N(t)+1} \geq x) \geq \sum_{n=0}^{\infty} \mathcal{F}(x) \mathbb{P}(N(t) = n) = \mathcal{F}(x). \]

When \( \mathcal{F}(x) = 1 - e^{-\lambda x} \), we have

\[ \mathbb{P}(X_{N(t)+1} \geq x) = \mathbb{P}(X_{1} \geq x, N(t) = 0) + \mathcal{F}(x) - \mathbb{P}(X_{1} \geq x) \mathbb{P}(N(t) = 0) \]
\[ = e^{-\lambda \max\{t,x\}} + e^{-\lambda x} - e^{-\lambda x} e^{-\lambda t}. \]

Problem 3.4

\[ m(t) = \mathcal{F}(t) + \sum_{n=2}^{\infty} F_{n}(t) \]
\[ = \mathcal{F}(t) + \sum_{n=2}^{\infty} \int_{0}^{t} F_{n-1}(t-x) d\mathcal{F}(x) \]
\[ = \mathcal{F}(t) + \int_{0}^{t} \sum_{n=2}^{\infty} F_{n-1}(t-x) d\mathcal{F}(x) \]
\[ = \mathcal{F}(t) + \int_{0}^{t} m(t-x) d\mathcal{F}(x). \]

Problem 3.5: Let

\[ \phi(F)(s) = \int_{0}^{\infty} e^{ist} d\mathcal{F}(t) \]

be the characteristic function of a random variable with density \( F \). So \( \phi(F_{n}) \) is the characteristic function of \( X_{1} + X_{2} + \cdots + X_{n} \). Because \( X_{i} \) are i.i.d, we have \( \phi(F_{n}) = [\phi(F)]^{n} \). From \( m(t) = \sum_{n=1}^{\infty} F_{n}(t) \) we obtain

\[ \phi(m) = \sum_{n=1}^{\infty} [\phi(F)]^{n} = \frac{\phi(F)}{1 - \phi(F)}. \]
Hence
\[ \phi(F) = \frac{\phi(m)}{1 + \phi(m)}. \]

In other words, \( \phi(F) \) is uniquely determined by \( m(t) \). Because a distribution is uniquely determined by its characteristic function, so \( F \) is uniquely determined by \( m(t) \).

**Note** You may use moment generating function instead of the characteristic function.

**Problem 3.7**
When \( F \) is a uniform distribution on \((0, 1)\), we have \( F(x) = x \) and \( dF(x) = dx \) for \( 0 < x < 1 \). Thus, Problem 3.4 gives
\[ m(t) = t + \int_0^t m(t - x)dx. \]

By changing variable \( t - x = s \) in the integral, we obtain
\[ m(t) = t + \int_0^t m(s)ds. \]

Taking derivative with respect to \( t \), we obtain
\[ m'(t) = 1 + m(t). \]

Solving the differential equation with the initial condition \( m(0) = 0 \), we obtain \( m(t) = e^t - 1 \).

For the second part, let \( K(t) = \min\{k : X_1 + X_1 + \cdots X_k > t\} \). Then \( K(t) = N(t) + 1 \). Thus
\[ \mathbb{E}K(1) = \mathbb{E}N(1) + 1 = m(1) + 1 = e - 1 + 1 = e. \]