Solution to HW 4

Problem 3.11.

(a) Let $X_i$ be the number of days of travel during the $i$-th trial. Let $N$ be the number of trials needed to get freedom, i.e. $N = \min\{n : X_n = 2\}$. Then $N$ is a stop time, and $T = \sum_{i=1}^{N} X_i$.

(b) $N$ is a geometric random variable with parameter $p = 1/3$. So, $\mathbb{E}(N) = 3$. $\mathbb{P}(X_1 = 2) = \mathbb{P}(X_1 = 4) = \mathbb{P}(X_1 = 8) = 1/3$. So, $\mathbb{E}(X_1) = 14/3$. Thus, $\mathbb{E}T = \mathbb{E}(N) \cdot \mathbb{E}(X_1) = 14$.

(c) Given $N = n$, the value of each $X_i$ is either 4 or 8 for $1 \leq i \leq n - 1$, and $X_n = 2$. For $0 \leq k \leq n - 1$,

$$\mathbb{P}(X_1 + \cdots + X_n = 4(n - 1 + k) + 2, N = n) = \binom{n-1}{k} \left( \frac{1}{3} \right)^n.$$ 

$$\mathbb{P}(X_1 + X_2 + \cdots + X_N = 4(n - 1 + k) + 2|N = n) = \binom{n-1}{k} \left( \frac{1}{3} \right)^n \left( \frac{2}{3} \right)^{n-1} \frac{1}{3} = \binom{n-1}{k} 2^{-n+1}.$$ 

Thus,

$$\mathbb{E}(X_1 + \cdots + X_n|N = n) = \sum_{k=0}^{n-1} (4n + 4k - 2) \binom{n-1}{k} 2^{-n+1} = 6n - 4.$$ 

A simple way to solve (c) is to notice that for $1 \leq i \leq n - 1$, $\mathbb{P}(X_i = 4|N = n) = \mathbb{P}(X_i = 8|N = n) = 1/2$. Thus, $\mathbb{E}(X_i|N = n) = 6$. Hence

$$\mathbb{E}(X - 1 + \cdots + X_N|N = n) = \sum_{i=1}^{n-1} \mathbb{E}(X_i|N = n) + 2 = 6(n - 1) + 2 = 6n - 4.$$ 

(d)

$$\mathbb{E}(X_1 + \cdots + X_N) = \sum_{n=1}^{\infty} (6n - 4)\mathbb{P}(N = n) = 6 \cdot 3 - 4 = 14.$$ 

Problem 2.

(a). Using Excel, we see that for $\mathbb{P}(N(20) \leq 14) = 0.916541527$, while $\mathbb{P}(N(20) \leq 15) = 0.951259597$. So, the 95 percentile of $N(20)$ is 15.

(b). The 95 percentile for $N(0,1)$ is 1.645. So, using approximation we have

$$\mathbb{P} \left( N(20) < \frac{t}{\mu} + 1.645\sigma \sqrt{\frac{t}{\mu^3}} \right) \approx 0.95,$$ 

here $t = 20$, $\mu = 2$ and $\sigma = 2$ (for exponential random variable $\sigma = \mu$). That is,

$$\mathbb{P} (N(20) < 15.2) \approx 0.95,$$ 

So the estimate 95 percentile of $N(20)$ is 15.
(c). For fixed integer $t$, the estimate 50 percentile of $N(t)$ is always the integer part of $t/\nu$; while the true 50 percentile of $N(t)$ is the smallest integer $n$ such that $\mathbb{P}(N(t) \leq n) \geq 0.5$.

For $t = 1$, the estimate 50 percentile of $N(1)$ is 0, the true percentile is also 0. ($\mathbb{P}(N(1) \leq 0) = 0.6065$)

For $t = 2$, the estimate 50 percentile of $N(2)$ is 1, the true percentile is also 1. ($\mathbb{P}(N(2) \leq 0) = 0.3679, \mathbb{P}(N(2) \leq 1) = 0.7357$.)

For $t = 3$, the estimate 50 percentile of $N(2)$ is 1, the true percentile is also 1. ($\mathbb{P}(N(3) \leq 0) = 0.22313016, \mathbb{P}(N(3) \leq 1) = 0.5578254$.)

For $t = 4$, the estimate 50 percentile of $N(2)$ is 2, the true percentile is also 2. ($\mathbb{P}(N(3) \leq 0) = 0.40600585, \mathbb{P}(N(3) \leq 2) = 0.676676416$.)

It seems that the estimate is always correct.

**Problem 3.12**

For non-lattice case, let

$$ h(s) = \begin{cases} 
1 & 0 \leq s \leq a \\
0 & s > a
\end{cases}. $$

For $t > 0$,

$$ \int_0^{t+a} h(t-s)dm(s) = \int_t^{t+a} 1dm(s) = m(t+a) - m(t). $$

Thus, by the key renewal theorem,

$$ \lim_{t \to \infty} [m(t+a) - m(t)] = \lim_{t \to \infty} \int_0^{t+a} h(t-s)dm(s) = \frac{1}{\mu} \int_0^{\infty} h(s)ds = \frac{1}{\mu} \int_0^{a} ds = \frac{a}{\mu}. $$

Note that the book contains a typo. The right hand side of the equation in the key renewal theorem should be

$$ \frac{1}{\mu} \int_0^{\infty} h(s)ds \not= \frac{1}{\mu} \int_0^{t} h(t)dt. $$

(Note: for lattice case, there is a lattice version of the key renewal theorem, which states as follows: if $X_i$ are lattice with period 1, then for any absolute convergent series $\sum_{n=0}^{\infty} b_n$,

$$ \lim_{m \to \infty} \sum_{k=0}^{m} b_{m-k}u(k) = \frac{1}{\mu} \sum_{n=0}^{\infty} b_n, $$

where $u(k)$ is the probability that there is a renewals at $k$. From this you can derive lattice case of Blackwell’s theorem. Since this part is not in the book, you don’t need to Problem 3.12 in the lattice case.)