Solution to HW 5

Problem 4.1.
If $i \geq s$, then for $0 \leq j < i$,
\[
\mathbb{P}(X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, ..., X_1 = i_1) = \mathbb{P}(\text{ demand } = i-j) = \alpha_{i-j};
\]
for $j = i$, we have
\[
\mathbb{P}(X_{n+1} = i|X_n = i, X_{n-1} = i_{n-1}, ..., X_1 = i_1) = \mathbb{P}(\text{ demand } = 0 \text{ or demand } > i) = \alpha_0 + \alpha_{i+1} + \alpha_{i+2} + \cdots;
\]
If $i < s$, then for $0 \leq j < S$,
\[
\mathbb{P}(X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, ..., X_1 = i_1) = \mathbb{P}(\text{ demand } = S-j) = \alpha_{S-j};
\]
for $j = S$,
\[
\mathbb{P}(X_{n+1} = i|X_n = i, X_{n-1} = i_{n-1}, ..., X_1 = i_1) = \mathbb{P}(\text{ demand } = 0 \text{ or demand } > S) = \alpha_0 + \alpha_{s+1} + \alpha_{s+2} + \cdots.
\]
Thus, $X_n$ is a Markov chain with transition matrix $(P_{ij})$ given above. (Note that the transition matrix satisfies $\sum_{j=0}^{\infty} P_{ij} = 1$)

Problem 4.2
Denote $P_{ij}^{(1)} = P_{ij}$, and for $l \geq 1$ define $P_{ij}^{(l+1)} = \sum_{k=0}^{\infty} P_{ik}^{(l)} P_{kj}^{(1)}$. We show that for all $m < n$,
\[
(1) \quad \mathbb{P}(X_n = j|X_m = i, X_{m-1} = i_{m-1}, ..., X_1 = i_1) = P_{ij}^{(n-m)}
\]
Indeed, when $m = n - 1$, (1) is just the definition of Markov chain. Suppose (1) holds for $m = l$, then
\[
\mathbb{P}(X_n = j|X_l = k, X_{l-1} = i_{l-1}, ..., X_1 = i_1) = P_{kj}^{(n-l)}.
\]
We show that (1) holds for $m = l - 1$. In fact,
\[
\mathbb{P}(X_n = j|X_{l-1} = i, X_{l-2} = i_{l-2}, ..., X_1 = i_1) = \sum_{k=0}^{\infty} \mathbb{P}(X_n = j|X_l = k, X_{l-1} = i, ..., X_1 = i_1) \mathbb{P}(X_l = k|X_{l-1} = i, ..., X_1 = i_1) = \sum_{k=0}^{\infty} P_{kj}^{(n-l)} P_{ik} = P_{ij}^{(n-l+1)}.
\]
Now that (1) holds for all $m$. Let $I$ be any subset of $\{1, 2, ..., m-1\}$. Denote $A = \{X_n = j\}$, $B = \{X_m = i, X_p = i_p, p \in I\}$ and $C = \{X_q = i_q, q \notin I, q < m\}$. Then
\[
\mathbb{P}(A|B, C) = \mathbb{P}(X_n = j|X_m = i, X_{m-1} = i_{m-1}, ..., X_1 = i_1) = P_{ij}^{(n-m)}.
\]
Thus,
\[
\mathbb{P}(A|B) = \sum_C \mathbb{P}(A|B, C) \mathbb{P}(C|B) = P_{ij}^{(n-m)} \sum_C \mathbb{P}(C|B) = P_{ij}^{(n-m)}.
\]
In particular, if \( m = n_k, I = \{n_1, n_2, ..., n_{k-1}\} \), we obtain,
\[
\mathbb{P}(A|B) = \mathbb{P}(X_n = j|X_{n_1} = i_{n_1}, ..., X_{n_k} = i) = P_{ij}^{(n-n_k)}.
\]
If \( m = n_k \) and \( I = \emptyset \), we obtain \( \mathbb{P}(X_n = j|X_{n_k} = i) = P_{ij}^{(n-n_k)} \).

**Problem 4.8**

(a) Let \( T_n \) be the time between \( n \)-th and \( (n+1) \)-th record. Let \( Y_1, Y_1, ... \), be the value of the random variable after \( n \)-th record. Then for \( j > i \),
\[
\mathbb{P}(R_{n+1} = j|R_n = i, R_{n-1} = i_{n_1}, ..., R_1 = i_1) = \sum_{k=1}^{\infty} \mathbb{P}(R_{n+1} = j, T_n = k|R_n = i, R_{n-1} = i_{n_1}, ..., R_1 = i_1) = \sum_{k=1}^{\infty} \mathbb{P}(Y_1 \leq i, ..., Y_{k-1} \leq i, Y_k = j) = \sum_{k=1}^{\infty} (\alpha_0 + \cdots + \alpha_i)^{k-1}\alpha_j = 1 - (\alpha_0 + \cdots + \alpha_i).
\]
For \( j \leq i \), \( \mathbb{P}(R_{n+1} = j|R_n = i, R_{n-1} = i_{n_1}, ..., R_1 = i_1) = 0 \). Thus, \( R_n \) is a Markov chain.

(b)
\[
\mathbb{P}(T_4 = 1|T_3 = 1, T_2 = 1) = \frac{\mathbb{P}(T_4 = 1, T_3 = 1, T_2 = 1)}{\mathbb{P}(T_3 = 1, T_2 = 1)} = \frac{\mathbb{P}(X_4 > X_3, X_3 > X_2, X_2 > X_1)}{\mathbb{P}(X_3 > X_2, X_2 > X_1)} = \frac{\mathbb{P}(X_4 > X_3 > X_2 > X_1)}{\mathbb{P}(X_3 > X_2 > X_1)}.
\]
\[
\mathbb{P}(T_4 = 1|T_3 = 1, T_2 = 2) = \frac{\mathbb{P}(T_4 = 1, T_3 = 1, T_2 = 2)}{\mathbb{P}(T_3 = 1, T_2 = 2)} = \frac{\mathbb{P}(X_5 > X_4, X_4 > X_3, X_3 > X_1 > X_2)}{\mathbb{P}(X_4 > X_3, X_3 > X_1 > X_2)} = \frac{\mathbb{P}(X_5 > X_4 > X_3 > X_1 > X_2)}{\mathbb{P}(X_4 > X_3 > X_1 > X_2)}.
\]
Thus, \( \mathbb{P}(T_4 = 1|T_3 = 1, T_2 = 1) \neq \mathbb{P}(T_4 = 1|T_3 = 1, T_2 = 2) \). So \( T_n \) is not a Markov chain.

Next, we show \( (R_n, T_n) \) is a Markov chain. For \( j > i \), let \( Y_1, Y_1, ... \), be the value of the random variable after \( n \)-th record.
\[
\mathbb{P}((R_{n+1}, T_{n+1}) = (j, p)|(R_n, T_n) = (i, q), (R_{n-1}, T_{n-1}) = (i_{n-1}, q_{n-1}), ..., (R_1, T_1) = (i_1, q_1)) = \mathbb{P}(Y_1 \leq i, Y_2 \leq i, ..., Y_{p-1} \leq i, Y_p = j) = (\alpha_0 + \cdots + \alpha_i)^{p-1}\alpha_j.
\]
Thus, \((R_n, T_n)\) is a Markov chain with transition matrix \(P_{(i,q),(j,p)} = (\alpha_0 + \cdots \alpha_i)^{p-1}\alpha_j\) for \(j > i\) and \(P_{(i,q),(j,p)} = 0\) for \(j \leq i\).

(c) Similarly, \(P_{n+1} = \sum_{j=0}^{n} P_{(n,j)}\).

Note that when \(X_i\) are continuous random variables \(P(X_i = X_j) = 0\). So, we can replace “≤” by “i” in the numerator above. Suppose there are \(m\) different ways of ordering \(X_1, \ldots, X_i\) so that \(S_n = i, S_{n-1} = i_{n-1}, \ldots, S_1 = i_1\). We see how many ways we can order \(X_1, \ldots, X_j\) so that \(X_j > X_i, X_{j-1} < X_i, \ldots, X_{i+1} < X_i, S_n = i_{n-1}, \ldots, S_1 = i_1\). For any given ordering of \(X_1, \ldots, X_i\), we put \(X_j\) before \(X_i\), (i.e. \(X_j > X_i\)) and inserting \(X_{j-1}, \ldots, X_{i+1}\) one by one anywhere after \(X_i\), without changing the order of \(X_1, \ldots, X_i\). Clearly, there is only one place to choose for \(X_j\), and \(i\) places for \(X_{j-1}\), \((i + 1)\) places for \(X_{j-2}\), \(\ldots\), and finally \((j - 2)\) places for \(X_{i+1}\). Thus, the total number of ordering \(X_1, \ldots, X_j\) is \(m \times i \times (i + 1) \cdots (j - 2)\). Hence

\[
P(S_n = i, S_{n-1} = i_{n-1}, \ldots, S_1 = i_1) = \frac{m}{i!}.
\]

Therefore

\[
P(S_{n+1} = j | S_n = i, S_{n-1} = i_{n-1}, \ldots, S_1 = i_1) = \frac{i}{j(j-1)}.
\]

That is, \(S_n\) is a Markov chain with transition matrix \(P_{ij} = \frac{i}{j(j-1)}\) for \(j > i\), and \(P_{ij} = 0\) for \(j \leq i\). (Note that \(\sum_{j=1}^{\infty} P_{ij} = \sum_{j=i+1}^{\infty} \frac{i}{j(j-1)} = i[(\frac{1}{1} - \frac{1}{i+1}) + (\frac{1}{i+1} - \frac{1}{i+2}) + \cdots] = 1\).

**Problem 4.10**

(a) For a specified non-infected individual to stay uninfected is that he does not contact any of the \(i\) infectious persons, which has probability \(1 - p)^i\). Thus, he will become infected with probability \(1 - (1 - p)^i\).

(b) \(P(X_{n+1} = j | X_n = i)\) depends on \(Y_n\). Indeed, if \(Y_n = 0\), then no one can be further infected, so \(P(X_{n+1} = j | X_n = i) = 0\). Otherwise, if \(Y_n \neq 0\), then clearly \(P(X_{n+1} = j | X_n = i) \neq 0\).

(c) Similarly, \(P(Y_{n+1} = j | Y_n = i)\) depends on \(X_n\). So \(Y_n\) is not a Markov chain either.

(d) Consider the event \(Y_n = j\). Suppose \(k\) of these \(j\) people will become infectious, then \(j - k\) people remain noninfected. Note that the probability

\[
P((X_{n+1}, Y_{n+1}) = (k, j-k) | (X_n, Y_n) = (i, j), (X_{n-1}, Y_{n}) = (i_{n-1}, j_{n-1}), \ldots, (X_1, Y_1) = (i_1, j_1))
\]
is the probability that $k$ of the $j$ noninfected persons are infected by the $i$ infectious people during the $(n+1)$-th time period. By (a), each of the noninfected gets infected with probability $1 - (1 - p)^i$. Therefore, $k$ of them get infected with probability

$$P_{(k,j-k),(i,j)} = \binom{j}{k} [1 - (1 - p)^i]^k [(1 - p)^i]^{j-k}.$$