Hint for 4.25 (c)

Let $e_{i,m}$ be the expected number of visits to state 5 in $m$ transitions, given the chain starts at state $i$. Then, $e_{3,7} = pe_{4,6} + qe_{2,6}; e_{4,6} = p(e_{5,5} + 1) + qe_{3,5}; e_{2,6} = pe_{3,5} + qe_{1,5}$. Thus,

(1) \[ e_{3,7} = p^2 + p^2 e_{5,5} + 2pq e_{3,5} + q^2 e_{1,5}. \]

Also note that $e_{5,5} = qe_{4,4} + pe_{6,4} = qe_{4,4} + 0; e_{3,5} = pe_{4,4} + qe_{2,4}; e_{1,5} = pe_{2,4} + qe_{0,4} = pe_{2,4} + 0$. Plugging into (1) we obtain:

(2) \[ e_{3,7} = p^2 + 3p^2 qe_{4,4} + 3pq^2 e_{2,4}. \]

Further notice that $e_{4,4} = p(e_{5,3} + 1) + qe_{3,3}; e_{2,4} = pe_{3,3} + qe_{1,3}$. Plugging into (2), we have

(3) \[ e_{3,7} = p^2 + 3p^3 q + 3p^3 qe_{5,3} + 6p^2 q^2 e_{3,3} + 3pq^3 e_{1,3}. \]

Now $e_{5,3} = qe_{4,2} + 0; e_{3,3} = pe_{4,2} + qe_{2,2}; e_{1,3} = 0 + pe_{2,2}$. Hence (3) gives

\[ e_{3,7} = p^2 + 3p^3 q + 9p^3 q^2 e_{4,2} + 9p^2 q^3 e_{2,2}. \]

It is easy to see that $e_{4,2} = p(e_{5,1} + 1) + qe_{3,1} = p$ and $e_{2,2} = 0$, so we finally have

\[ e_{3,7} = p^2 + 3p^3 q + 9p^4 q^2. \]