Review Problems for Test 3

1. In a branching process the number of offspring per individual has a binomial distribution with parameters 2, \( p \). Starting with a single individual, calculate
   (a) the extinction probability;
   (b) the probability that the population becomes extinct for the first time in the third generation.

   **Solution:** \( P_0 = (1 - p)^2, P_1 = 2p(1 - p), P_2 = p^2 \) and \( P_i = 0 \) for \( i > 2 \).
   (a) Because \( P_0 > 0 \) and \( P_1 + P_0 < 1 \), by Theorem 4.5.1. \( \pi_0 \) is the smallest positive solution of \( x = P_0 + P_1x + P_2x^2 \). That is \( x = (1 - p)^2 + 2p(1 - p)x + p^2x \). Solve the equation, we have

   \[
   \pi_0 = \begin{cases} 
   1 & p < \frac{1}{2} \\
   1 - \frac{2p - 1}{p^2} & p > \frac{1}{2} 
   \end{cases}.
   \]

   (b) Let \( X_n \) be the population of \( n \)-th generation. Then for the population becomes extinct for the first time in the third generation, \((X_1, X_2, X_3)\) has to be one of the following: \((1, 1, 0), (1, 2, 0), (2, 1, 0), (2, 2, 0), (2, 3, 0)\) and \((2, 4, 0)\). These are mutually exclusive events. There probability are respectively \([P_1 \cdot P_1 \cdot P_0], \ [P_1 \cdot P_2 \cdot P_0^2], \ [P_2 \cdot 2pP_1 \cdot P_0], \ [P_2 \cdot (2P_0P_2 + P_1^2) \cdot P_0^2], \ [P_2 \cdot 2P_1P_2 \cdot P_0^3]\) and \([P_2 \cdot P_2^2 \cdot P_0^6]\). Hence the probability that the population becomes extinct for the first time in the third generation the sum of all these six probabilities, which simplifies to

   \[
   \]

2. A Markov chain on the line with state 0, 1, ..., \( n \). If moves from state 0 to state 1 with probability \( 1 \). If it is in state \( i \), \( 1 \leq i < n \), it moves to \( i + 1 \) with probability \( p \) and move to \( i - 1 \) with probability \( 1 - p \). If it is in state \( n \), it move to \( n - 1 \) with probability 1. Suppose the chain starts at 0. Find the expected number of transition it takes to visit all \( n + 1 \) states.

   **Solution:** After the chain enters state 1, By Gambler’s ruin problem it take expected number \( \mathbb{E}B \) (see page 188 with \( i = 1 \)) to reach either \( n \) or 0, and with probability \( \alpha \) (see page 188 with \( i = 1 \)) it enters \( n \) before returning to 0. If it comes back to 0 before entering \( n \), then process start again. Therefore the number of trials is a geometric random variable with parameter \( \alpha \) (mean \( 1/\alpha \)). Hence the expected number of steps it takes to reach \( n \) is \( (\mathbb{E}B + 1)/\alpha \).

3. A particle moves among \( n \) locations that are arranged in a circle (so that \( n - 1 \) and 1 are the two neighbors of \( n \)). At each step it moves one position either clockwise with probability \( p \) or counterclockwise position with probability \( 1 - p \). Find the transition probabilities of the reverse chain. Is the chain time reversible?

   **Solution:** \( P_{i,i+1} = p \) and \( P_{i,i-1} = 1 - p \) for \( 1 < i < n \), and \( P_{1,2} = p, P_{n,1} = p, P_{n,n-1} = 1 - p \). It is easy to check that \( \pi_i = 1/n \) is a stationary distribution. Thus, \( P_{ij}^* = P_{ji} \).

   The chain is time reversible if and only if \( 1/1/2 \).

4. For Markov chain in Example 4.3(D) on page 182, show that it is a time reversible Markov chain.
**Solution:** Note that

\[ P_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \sum_{k=0}^{i} \binom{i}{k} (1-p)^k p^{i-k} \cdot \frac{e^{-\lambda \lambda j - k}}{(j-k)!}. \]

Thus by the formula of \( \pi_i \) on page 182, it is ready to check that \( \pi_i P_{ij} = \pi_j P_{ji} \).

5. Consider the gambler’s ruin problem with \( p = 0.6 \) and \( n = 4 \). Starting in state 2. Find the
(a) the expected number of visits to state 3.
(b) the probability of ever visiting state 1.

**Solution:**

\[
Q = \begin{pmatrix}
0 & 0.6 & 0 \\
0.4 & 0 & 0.6 \\
0 & 0.4 & 0
\end{pmatrix}, \quad (I - Q)^{-1} = \begin{pmatrix}
1.4615 & 1.1538 & 0.6923 \\
0.7692 & 1.9231 & 1.1538 \\
0.3077 & 0.7692 & 1.4615
\end{pmatrix}.
\]

So (a) the expected number of visits to state 3 is 1.1538; (b) the probability of ever visiting state 1 is 0.7692/1.4615 = 0.5263.

6. Prove that in an irreducible Markov chain with stationary distribution \( \{\pi_i\} \), \( \pi_j \geq P_{ij}^k \pi_i \) for any \( k \geq 1 \).

**Solution:** If \( \{\pi\} \) is a stationary distribution, then

\[
\pi_j = \sum_l \pi_l P_{lj} = \sum_i \pi_i P_{il} P_{lj} = \sum_i \sum_l \pi_i P_{il} P_{lj} = \sum_i \pi_i P_{lj}^2 = \cdots = \sum_i \pi_i P_{ij}^k \geq \pi_1 P_{ij}^k.
\]

Note that this implies \( \mu_{jj} \leq \mu_{ii} / P_{ij}^k \).