Solution to Review 1

1. Let $N(t)$ be a Poisson process with mean $\lambda$. Prove or disprove that
   (i) $N_1(t) := N(t+1) - N(1)$ is a Poisson process with rate $\lambda$;
   (ii) $N^*(t) := N(2t) - N(t)$ is a Poisson process with mean $\lambda$.

Solution $N_1(0) = 0$; $N_1(b) - N_1(a) = N(b) - N(a)$ is independent of $N_1(d) - N_1(c) = N(d) - N(c)$ when $(a, b]$ and $(c, d]$ are disjoint. Hence $N_1$ has independent increment;

$$P(N_1(t+s) - N_1(s) = n) = P(N(t+s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda s)^n}{n!}.$$

Hence $N_1(t)$ is a Poisson process with rate $\lambda$.

$N^*(t)$ is far from being a Poisson process. For example, $N^*(t)$ does not have stationary increment. Indeed

$$P(N^*(1) = 0) = P(N(2) - N(1) = 0) = P(N(1) = 0) = e^{-\lambda}.$$

While

$$P(N^*(2) - N^*(1) = 0) = P(N(4) - N(2) = N(2) - N(1))$$

$$= \sum_{n=0}^{\infty} P(N(4) - N(2) = n) P(N(2) - N(1) = n)$$

$$= \sum_{n=0}^{\infty} \left( \frac{e^{-2\lambda}(2\lambda)^n}{n!} \right) \left( \frac{e^{-\lambda}(\lambda)^n}{n!} \right).$$

2. Suppose cars pass Third street according to a Poisson process with a rate of 10 cars per minute. Frank wants to cross the street. He waits until there is no car come in the next 5 seconds. What is his expected waiting time?

Solution: Let $s = \frac{5}{60}$ minute (5 seconds). The waiting time can be expressed as

$$W = X_11_{X_1<s} + X_21_{X_2<s}1_{X_1<s} + X_31_{X_3<s}1_{X_2<s}1_{X_1<s} + \cdots.$$

$$E[W] = EX_11_{X_1<s} + EX_21_{X_2<s}1_{X_1<s} + EX_31_{X_3<s}1_{X_2<s}1_{X_1<s} + \cdots$$

$$= \int_0^s 10te^{-10t} dt + \int_0^s 10te^{-10t} dt \cdot e^{-10s} + \int_0^s 10te^{-10t} dt \cdot (e^{-10s})^2 + \cdots$$

$$= \int_0^s 10te^{-10t} dt \cdot \frac{1}{1 - e^{-10s}}$$

$$= \frac{1}{10} - \frac{s}{e^{10s} - 1}.$$

3. Let $S_n$ be the time of $n$th event of the Poisson process $N(t)$. Find $E[S_i|N(t) = n]$ (Hint: consider two different cases separately: $i \leq n$ and $i > n$.)
Solution: If $i \leq n$, then $$\mathbb{E}[S_i|N(t) = n] = \mathbb{E}[X_1 + X_2 + \cdots + X_i|N(t) = n] = i\mathbb{E}[X_1|N(t) = n] = \frac{it}{2}.$$ If $i > n$, then $$\mathbb{E}[S_i|N(t) = n] = \frac{nt}{2} + \mathbb{E}[X_{n+1} + X_{n+2} + \cdots + X_i|N(t) = n] = \frac{tn}{2} + \frac{i-n}{\lambda}.$$ 4. Bus arrives the stop at the intersection of Line street and Sixth street according to a non-homogeneous Poisson process with intensity function $\lambda(t)$. Frank decides to wait for a bus for up to time $s$. Find Frank’s expected waiting time.

Solution: Let $W$ be the waiting time, then $W = X_11_{X_1<s} + s1_{X_1\geq s}$. Thus, $$\mathbb{E}W = \mathbb{E}X_11_{X_1<s} + s\mathbb{P}(X_1 \geq s) = \int_0^s t\lambda(t)e^{-m(t)}dt + se^{-m(s)},$$ where $m(t) = \int_0^t \lambda(x)dx$.

5. Suppose that cars cars pass a certain checkpoint on a highway according to a Poisson process with rate 60 cars per minute. The number of passengers in the cars are independent with the common distribution: $\mathbb{P}(Y = 1) = 0.5$, $\mathbb{P}(Y = 2) = 0.3$, $\mathbb{P}(Y = 3) = 0.15$ and $\mathbb{P}(Y = 4) = 0.05$. A car is full if it has four passengers.

(i) Find the expected number of passengers that pass in the next minute.
(ii) Find the probability that two full cars pass in the next 10 seconds.

Solution: (i) $$\mathbb{E} \sum_{i=1}^N (1)Y_i = \mathbb{E}N(1) \cdot \mathbb{E}Y = 60 \cdot 1.75 = 105.$$ (ii) Full cars come according to the Poisson process with rate $\lambda p = 60 \cdot 0.05 = 3$ per minute. Thus the expected number in next 10 seconds (1/6 minutes) is 0.5, and the probability that two full cars come is $e^{-0.5}(0.5)^2/2 = e^{-0.5}/8$.

6. Consider a renewal process $N(t)$ whose interarrival times $X_1$, $X_2$, ... are i.i.d with pdf $f(x)$. Denote $m(t) = \mathbb{E}[N(t)]$ and $v(t) = \mathbb{E}[N(t)^2]$. By conditioning on $X_1$, prove that $$v(t) = \int_0^t [1 + 2m(t-x) + v(t-x)]f(x)dx.$$ Solution

$$v(t) = \mathbb{E}[N(t)^2] = \int_0^t \mathbb{E}[N(t)^2|X_1 = s]dF(s)$$

$$= \int_0^t \mathbb{E}[(N(t) - N(s) + 1)^2|X_1 = s]dF(s)$$

$$= \int_0^t \mathbb{E}[(N(t-s) + 1)^2]dF(s)$$

$$= \int_0^t \mathbb{E}[N(t-s)^2 + 2N(t-s) + 1]dF(s)$$

$$= \int_0^t [v(t-s) + 2m(t-s) + 1]dF(s).$$