Combinatorial Dimension in Fractional Cartesian Products

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September 11, 2004

Abstract

The combinatorial dimension relative to an arbitrary fractional Cartesian product is defined. Relations between dimensions in certain archetypal instances are derived. Random sets with arbitrarily prescribed dimensions are produced; in particular, scales of combinatorial dimension are shown to be continuously and independently calibrated. A combinatorial concept of cylindricity is key.

1 Definitions, statement of problem

The idea of a *fractional* Cartesian product and a subsequent measurement of combinatorial dimension appeared first in a harmonic-analytic context in the course of filling "analytic" gaps between successive (ordinary) Cartesian products of spectral sets [2, 3]. (Detailed accounts of this, and much more, appear in [5].)

Succinctly put, combinatorial dimension is an index of interdependence. Attached to a subset of an ordinary Cartesian product, it gauges precisely the interdependence of restrictions to the set, of the canonical projections from the Cartesian product onto its independent coordinates. We can analogously gauge the interdependence of restrictions to the same set, of projections from the Cartesian product onto interdependent coordinates of a prescribed *fractional* Cartesian product. We thus obtain distinct indices of interdependence associated, respectively, with distinct fractional Cartesian products. A question naturally arises: what are the relationships between these various indices?

To make matters precise, we first recall, and then extend basic notions found in Chapters XII and XIII of [5]. Let $E_1, \ldots, E_n$ be sets, and let $F \subset E_1 \times \cdots \times E_n$. (We refer to $E_1 \times \cdots \times E_n$ as the ambient product of F.) For integers $s > 0$ define

$$\Psi_F(s) = \max\big\{|F \cap (A_1 \times \ \cdots \times A_n| : A_i \subset E_i,\, |A_i| \leq s,\, i \in [n]\big\}, \tag{1}$$

where $[n] = \{1, \ldots, n\}$. For $a > 0$, define

$$d_F(a) = \sup\big\{\Psi_F(s)/s^a : s = 1, 2, \ldots\big\}. \tag{2}$$

The *combinatorial dimension* of F is

$$\dim F = \sup\{a : d_F(a) = \infty\} = \inf\{a : d_F(a) < \infty\}. \tag{3}$$

Next we define the fractional Cartesian products. For $S \subset [n]$, let π_S denote the canonical projection from $E_1 \times \cdots \times E_n$ onto the product whose coordinates are indexed by S,

$$\pi_S(\mathbf{y}) = (y_i : i \in S), \quad \mathbf{y} = (y_1, \ldots, y_n) \in E_1 \times \cdots \times E_n.$$

Let $\mathrm{U} = (S_1, \ldots, S_m)$ be a cover of $[n]$ (i.e., $S_1 \subset [n], \ldots, S_m \subset [n]$, and $\bigcup_{j=1}^m S_j = [n]$), and define a *fractional Cartesian product based on* U to be

$$(E_1 \times \cdots \times E_n)_{\mathrm{U}} = \big\{(\pi_{S_1}(\mathbf{y}), \ldots, \pi_{S_m}(\mathbf{y})) : \mathbf{y} \in E_1 \times \cdots \times E_n\big\}.$$

We view $(E_1 \times \cdots \times E_n)_{\mathrm{U}}$ as a subset of $E^{S_1} \times \cdots \times E^{S_m}$, and measure its combinatorial dimension by solving a linear programming problem ([4, 8]): if $E_1, \ldots, E_n$ are infinite sets, and

$$\alpha_{\mathrm{U}} = \max\left\{\sum_{i=1}^n x_i : x_i \geq 0, \ \sum_{i \in S_j} x_i \leq 1 \text{ for } j \in [m]\right\},$$

then

$$\dim(E_1 \times \cdots \times E_n)_{\mathrm{U}} = \alpha_{\mathrm{U}}. \tag{4}$$

Examples

1. (*Maximal and minimal fractional Cartesian products*) For integers $1 \leq k \leq n$, let U be a cover that is an enumeration of all $k-$subsets of $[n]$, and let V be a cover of $[n]$, all of whose elements are $k-$subsets of $[n]$ such that for every $i \in [n]$, $\mid \{S \in \mathrm{V} : i \in S\} \mid = k$. Then, $\alpha_{\mathrm{U}} = \alpha_{\mathrm{V}} = n/k$, and (taking $E_1 = \cdots = E_n = \mathbb{N}$) we obtain from (4)

$$\dim(\mathbb{N}^n)_{\mathrm{U}} = \dim(\mathbb{N}^n)_{\mathrm{V}} = \frac{n}{k}.$$

   The ambient product of $(\mathbb{N}^n)_{\mathrm{U}}$ is $\binom{n}{k}$-dimensional, and the ambient product of $(\mathbb{N}^n)_{\mathrm{V}}$ is n-dimensional. Generally, if k and n are relatively prime, then the dimension of the ambient product of any $\frac{n}{k}$-dimensional fractional Cartesian product is at least n, and no greater than $\binom{n}{k}$.

2. (*Random constructions*) Fractional Cartesian products are subsets of ambient products typically of high dimension. To wit, there are no non-trivial fractional Cartesian products in $\mathbb{N}^2$, whereas for arbitrary $\alpha \in (1, 2)$ there exists an abundance of *random* sets $F \subset \mathbb{N}^2$ with $\dim F = \alpha$ [6, 7]. How to produce deterministically $F \subset \mathbb{N}^2$ with $\dim F = \alpha$, where $\alpha \in (1, 2)$ is arbitrary, is an open problem.

3. *(An application)* From $\alpha_{\mathrm{U}} = 3/2$ in the archetypal case $n = 3$, $\mathrm{U} = (\{1,2\},\{2,3\},\{1,3\})$, we obtain that that the number of "triangles" formed from s given edges in a graph is less than $s^{\frac{3}{2}}$. More generally, following a canonical identification of a complex on n vertices with a cover U of $[n]$, we obtain from (4) that the number of complexes formed from s given simplices is bounded by $s^{\alpha_{\mathrm{U}}}$. These results are closely related to the Kruskal-Katona Theorem (drawn to our attention by Joel Spencer); see [10, 9].

Definition 1. *For $F \subset E_1 \times \cdots \times E_n$, and a cover $\mathrm{U} = (S_1, \ldots, S_m)$ of $[n]$, let*

$$F_{\mathrm{U}} = \big\{(\pi_{S_1}(\boldsymbol{y}), \ldots, \pi_{S_m}(\boldsymbol{y})) : \boldsymbol{y} \in F\big\},$$

and write

$$\dim_{\mathrm{U}} F = \dim F_{\mathrm{U}}. \tag{5}$$

We consider $\dim_{\mathrm{U}}$ as a gauge of interdependence of $\pi_{S_1}|_F, \ldots, \pi_{S_m}|_F$ (projections π_{S_i} restricted to F). By (4), for infinite F,

$$1 \leq \dim_{\mathrm{U}} F \leq \alpha_{\mathrm{U}}. \tag{6}$$

We continue to write $\dim F$ in the extremal case $\mathrm{U} = (\{1\}, \ldots, \{n\})$.

Problem 2. *Let $E_1, \ldots, E_n$ be infinite sets, and let U be a cover of $[n]$. For $F \subset E \times \ldots \times E_n$, what are the relations between $\dim F$ and $\dim_{\mathrm{U}} F$?*

In this article we analyze the first nontrivial case: $n = 3$, and $\mathrm{U} = (S_1, S_2, S_3)$, where $S_1 = \{1,2\}$, $S_2 = \{2,3\}$, $S_3 = \{1,3\}$. Throughout, for convenience (and with no loss of generality), we take $E_1 = E_2 = E_3 = \mathbb{N}$.

2 General bounds

Theorem 3. *If $F \subset \mathbb{N}^3$ and $\dim F \geq 2$, then*

$$\frac{\dim F}{2} \leq \dim_{\mathrm{U}} F \leq \frac{2 \dim F}{\dim F + 1}. \tag{7}$$

We will prove the right-side inequality in (7) (the non-trivial part of the theorem) by the use of Littlewood-type inequalities in fractional dimensions.

Let $E_1, \ldots, E_d$ be sets. Consider scalar d-tensors on $E_1 \times \cdots \times E_d$ with finite support

$$b = (b_{\mathbf{x}} : \mathbf{x} = (x_1, \ldots, x_d) \in E_1 \times \cdots \times E_d),$$

and define (the d-fold injective tensor-norm of b)

$$\|b\|_{\check{\otimes}_d} = \left\| \sum_{\mathbf{x} \in E_1 \times \cdots \times E_d} b_{\mathbf{x}} r_{x_1} \otimes \cdots \otimes r_{x_d} \right\|_{\infty}, \tag{8}$$

where $r_{x_1}, \ldots, r_{x_d}$ are Rademacher functions indexed by $E_1, \ldots, E_d$, whose respective domains are $\{-1,1\}^{E_1}, \ldots, \{-1,1\}^{E_d}$; e.g., p. 19 in [5]. For $F \subset E_1 \times \cdots \times E_d$, and $t \geq 1$, define

$$\zeta_F(t) = \sup \left\{ \left(\sum_{\mathbf{x}=(x_1,\ldots,x_d)\in F} |b_{\mathbf{x}}|^t \right)^{\frac{1}{t}} : \|b\|_{\check{\otimes}_d} \leq 1 \right\}. \tag{9}$$

The relation between the harmonic-analytic measurement ζ_F and the combinatorial measurement d_F is

$$\zeta_F(t) < \infty \iff d_F\left(\frac{t}{2-t}\right) < \infty, \ \ t \in [1,2). \tag{10}$$

(See Th. XIII.20 in [5].) To prepare for an application of (10), we define

$$\|b\|_{\check{\otimes}_{\mathrm{U}}} = \left\| \sum_{i,j,k} b_{ijk} r_{ij} \otimes r_{jk} \otimes r_{ik} \right\|_{\infty}, \tag{11}$$

for $b = (b_{ijk} : (i,j,k) \in \mathbb{N}^3)$ with finite support. We observe

$$\mathbf{1}_{(\mathbb{N}^3)_{\mathrm{U}}}(i_1,i_2,i_3,i_4,i_5,i_6) = \mathbb{E} r_{i_1}(\omega_1) r_{i_2}(\omega_2) r_{i_3}(\omega_2) r_{i_4}(\omega_3) r_{i_5}(\omega_1) r_{i_6}(\omega_3),$$

where $\mathbf{1}$ denotes *indicator function*, and $\mathbb{E}$ denotes *expectation* over $\omega_1, \omega_2, \omega_3$.

From this we deduce (an instance of Corollary XIII.7 in [5])

Lemma 4. *If* $\tilde{b} = (\tilde{b}_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2)$ *is a real-valued* 3*-tensor with finite support, and*

$$b_{\mathbf{j}} = \tilde{b}_{\pi_{S_1}(\mathbf{j})\pi_{S_2}(\mathbf{j})\pi_{S_3}(\mathbf{j})}, \ \ \mathbf{j} \in \mathbb{N}^3,$$

then

$$\|b\|_{\check{\otimes}_{\mathrm{U}}} \leq \|\tilde{b}\|_{\check{\otimes}_3}. \tag{12}$$

We require also a decomposition property, which follows from Lemma XIII.21 in [5]: if $\varphi \in l^\infty(\mathbb{N}^n)$, and

$$c_n(\varphi) = \sup \left\{ \frac{1}{s} \sum_{\mathbf{i}\in A_1\times\cdots\times A_n} |\varphi(\mathbf{i})| : s \in \mathbb{N}, A_v \subset \mathbb{N}, |A_v| = s, v \in [n] \right\},$$

then there exists a partition $\{Q_1, \ldots, Q_n\}$ of $\mathbb{N}^n$ such that for $v \in [n]$,

$$\sup_{k\in\mathbb{N}} \sum_{\mathbf{i}\in\pi_v^{-1}\{k\}} |\varphi(\mathbf{i})| \mathbf{1}_{Q_v}(\mathbf{i}) \ \leq \ c_n(\varphi),$$

where π_v is the v^{th} canonical projection from $\mathbb{N}^n$ onto $\mathbb{N}$.

Lemma 5. *If* $\varphi \in l^\infty(\mathbb{N}^n)$ *is supported in* $F \subset \mathbb{N}^n$*, then there exists a partition* $\{Q_1, \ldots, Q_n\}$ *of* $\mathbb{N}^n$ *such that for* $v \in [n]$*, and* $p \geq 1$*,*

$$\sup_{k\in\mathbb{N}} \sum_{\mathbf{i}\in\pi_v^{-1}\{k\}} |\varphi(\mathbf{i})|\mathbf{1}_{Q_v}(\mathbf{i}) \;\leq\; d_F(p)^{\frac{1}{p}} \|\varphi\|_q, \tag{13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For $s \in \mathbb{N}$, and s-sets $A_v \subset \mathbb{N}$ (i.e. $|A_v| = s$), $v \in [n]$, we estimate (by Hölder's inequality)

$$\frac{1}{s} \sum_{\mathbf{i}\in A_1\times\cdots\times A_n} |\varphi(\mathbf{i})| \leq \frac{1}{s} |F \cap (A_1 \times \ldots \times A_n)|^{\frac{1}{p}} \|\varphi\|_q.$$

Then $c_n(\varphi) \leq d_F(p)^{\frac{1}{p}} \|\varphi\|_q$, and the lemma follows from the decomposition property above. □

Lemma 6. *If* $F \subset \mathbb{N}^3$*,* $a \geq 2$*, and* $d_F(a) < \infty$*, then*

$$\zeta_{F_{\mathrm{U}}}\left(\frac{4a}{3a+1}\right) < \infty. \tag{14}$$

Proof. Following Lemma 4 and the definition of $\zeta_{F_{\mathrm{U}}}(a)$, we need to verify that if $b = (b_{ijk} : (i,j,k) \in \mathbb{N}^3)$ is a scalar 3-tensor with finite support, and $\|b\|_{\check{\otimes}_{\mathrm{U}}} \leq 1$, then

$$\sum_{(i,j,k)\in F} |b_{ijk}|^{\frac{4a}{3a+1}} \leq K,$$

where K depends only on F and a. To this end we use duality, and the "fractional" mixed norm inequalities

$$\sum_{i,j}\Big(\sum_k |b_{ijk}|^2\Big)^{\frac{1}{2}} \leq \lambda,\; \sum_{i,k}\Big(\sum_j |b_{ijk}|^2\Big)^{\frac{1}{2}} \leq \lambda,\; \sum_{j,k}\Big(\sum_i |b_{ijk}|^2\Big)^{\frac{1}{2}} \leq \lambda, \tag{15}$$

where $\lambda > 1$ is an absolute constant; see Lemma XII.1 in [5].

Suppose θ is an element in the unit ball of $l^{\frac{4a}{a-1}}(\mathbb{N}^3)$, and that θ is supported in F. Apply Lemma 5 with $p = a$ and $\varphi = |\theta|^4$, thus obtaining a partition $\{Q_1, Q_2, Q_3\}$ of $\mathbb{N}^3$, such that for $v = 1, 2, 3$,

$$\sup_{l\in\mathbb{N}} \sum_{\mathbf{i}\in\pi_v^{-1}\{l\}} |\theta(\mathbf{i})|^4\mathbf{1}_{Q_v}(\mathbf{i}) \;\leq d_F(a)^{\frac{1}{a}}.$$

For $l \in \mathbb{N}$, and $v = 1, 2, 3$, let $F_{lv} = F \cap Q_v \cap \pi_v^{-1}\{l\}$. For convenience, designate $T_1 = \{2,3\}$, $T_2 = \{1,3\}, T_3 = \{1,2\}$. For each l and v, apply Lemma 5 with $p = 2$, $F = F_{lv}$ (viewed as a subset of $\mathbb{N}^2$), and $\varphi = |\theta|^2 \cdot \mathbf{1}_{F_{lv}}$ (viewed as a function on $\mathbb{N}^2$), thus obtaining partitions $\{P_{l1v}, P_{l2v}\}$ of F_{lv}, such that

$$\begin{aligned} \sup_{l\in\mathbb{N}, i\in\mathbb{N}} \sum_{j\in\mathbb{N}}((|\theta|^2 \cdot \mathbf{1}_{P_{l1v}}) \circ \pi_{T_v})(i,j) \quad &\leq \quad d_F(a)^{\frac{1}{2a}}, \\ \sup_{l\in\mathbb{N}, i\in\mathbb{N}} \sum_{i\in\mathbb{N}}((|\theta|^2 \cdot \mathbf{1}_{P_{l2v}}) \circ \pi_{T_v})(i,j) \quad &\leq \quad d_F(a)^{\frac{1}{2a}}. \end{aligned} \tag{16}$$

(In applying here Lemma 5, we used $d_{F_{lv}}(2) \leq 1$.) For $u = 1, 2$, and $v = 1, 2, 3$, let

$$P_{uv} = \bigcup_{l \in \mathbb{N}} P_{luv},$$

and then write

$$\sum_{\mathbf{i} \in \mathbb{N}^3} b_{\mathbf{i}} \cdot \theta(\mathbf{i}) \; = \sum_{u \in \{1,2\}, v \in \{1,2,3\}} \Big(\sum_{\mathbf{i} \in \mathbb{N}^3} b_{\mathbf{i}} \cdot \theta(\mathbf{i}) \cdot \mathbf{1}_{P_{uv}}(\mathbf{i}) \Big). \tag{17}$$

By applying to each of the six sums on the right side of (17) the mixed-norm inequalities (15), together with (16) (via Cauchy-Schwarz and Hölder), we obtain

$$\Big| \sum_{\mathbf{i} \in \mathbb{N}^3} b_{\mathbf{i}} \cdot \theta(\mathbf{i}) \Big| \; \leq \; 6 \lambda d_F(a)^{\frac{1}{4a}}.$$

Therefore (by duality and definition of $\zeta_{F_{\mathrm{U}}}$),

$$\zeta_{F_{\mathrm{U}}} \left(\frac{4a}{3a+1} \right) \leq 6 \lambda d_F(a)^{\frac{1}{4a}}.$$

□

Proof of Theorem 3. Let A, B, C be arbitrary s-subsets of $\mathbb{N}$. Then

$$\begin{aligned} \Psi_{F_{\mathrm{U}}}(s^2) \quad &\geq \quad |F_{\mathrm{U}} \cap (A \times B) \times (B \times C) \times (A \times C)| \\ &= \quad |F \cap (A \times B \times C)|. \end{aligned}$$

Maximizing over A, B, C, we obtain

$$\Psi_{F_{\mathrm{U}}}(s^2) \geq \Psi_F((s),$$

which implies

$$d_{F_{\mathrm{U}}}(\frac{a}{2}) \geq d_F(a), \quad a > 0,$$

and hence the left inequality in (7).

If $a > \dim F$, then $d_F(a) < \infty$. By Lemma 6, $\zeta_{F_{\mathrm{U}}}(\frac{4a}{3a+1}) < \infty$. By (10), $d_{F_{\mathrm{U}}}(\frac{2a}{a+1}) < \infty$, which implies $\dim F_{\mathrm{U}} \leq \frac{2a}{a+1}$, and hence the right-side inequality in (7). □

3 Cylindrical sets

For $F \subset \mathbb{N}^3$, $v = 1, 2, 3$, and $k \in \mathbb{N}$, let

$$F_v(k) = \pi_v^{-1}\{k\} \cap F.$$

Also, for $E \subset \mathbb{N}^2$ and $k \in \mathbb{N}$, we denote

$$\begin{aligned} E_{1,k} \quad &= \quad \{(j : (j, k) \in E\}, \\ E_{2,k} \quad &= \quad \{(j : (k, j) \in E\}. \end{aligned}$$

Definition 7. *$F \subset \mathbb{N}^3$ is cylindrical in direction v ($v = 1, 2, 3$) if for all $a \geq 2$*

$$d_F(a) < \infty \;\Rightarrow\; \sup\{d_{F_v(k)}(a-1) : k \in \mathbb{N}\} < \infty.$$

F is cylindrical if it is cylindrical in at least one direction, and doubly-cylindrical if it is cylindrical in at least two directions.

We say $F \subset \mathbb{N}^3$ is a *cylinder* if $F = E \times H$ for $E \subset \mathbb{N}^2$ (base) and $H \subset \mathbb{N}$ (height). For such F, $\dim F = \dim E + \dim H$. Note also that cylinders are obviously cylindrical, but cylindrical sets need not be cylinders.

Theorem 8. **(i)** *If $F \subset \mathbb{N}^3$ is cylindrical with $\dim F \geq 2$, then*

$$\frac{\dim F}{2} \leq \dim_{\mathrm{U}} F \leq 2 - \frac{1}{\dim F - 1}. \tag{18}$$

(ii) *If $F \subset \mathbb{N}^3$ is a cylinder with infinite base and infinite height, then*

$$\dim_{\mathrm{U}} F = 2 - \frac{1}{\dim F - 1}. \tag{19}$$

(iii) *If $F \subset \mathbb{N}^3$ is doubly-cylindrical with $\dim F \geq 2$, then*

$$\dim_{\mathrm{U}} F = \frac{\dim F}{2}. \tag{20}$$

Proof. **(i)** The left inequality always holds (Theorem 3). We proceed to verify the inequality on the right, under the assumption that F is cylindrical in direction 3. Suppose $a > \dim F \geq 2$, and

$$\sup_{k \in \mathbb{N}} d_{F_3(k)}(a-1) \leq K < \infty. \tag{21}$$

Let A, B, C be arbitrary s-subsets of $\mathbb{N}^2$, and define

$$H = \left\{k : \max\{|B_{1,k}|, |C_{1,k}|\} \geq s^{\frac{1}{a-1}}\right\}.$$

Then, $|H| \leq 2s^{(a-2)/(a-1)}$, and

$$\sum_{k \in H} \sum_{i,j} \mathbf{1}_F(i,j,k)\mathbf{1}_A(i,j)\mathbf{1}_B(j,k)\mathbf{1}_C(i,k) \leq \sum_{k \in H} |A| \leq 2s^{\frac{2a-3}{a-1}}. \tag{22}$$

From (21) we obtain

$$\begin{aligned}
&\sum_{k \notin H} \sum_{i,j} \mathbf{1}_F(i,j,k)\mathbf{1}_A(i,j)\mathbf{1}_B(j,k)\mathbf{1}_C(i,k) \\
&\qquad \leq \sum_{k \notin H} \sum_{i,j} \mathbf{1}_F(i,j,k)\mathbf{1}_B(j,k)\mathbf{1}_C(i,k) \\
&\qquad \leq \sum_{k \notin H} d_{F_3(k)}(a-1) \cdot \max\{|B_{1,k}|^{a-1}, |C_{1,k}|^{a-1}\} \\
&\qquad \leq K s^{\frac{a-2}{a-1}} \sum_{k \notin H} (|B_{1,k}| + |C_{1,k}|) \\
&\qquad \leq 2K s^{\frac{2a-3}{a-1}}.
\end{aligned}$$

Combining this with (22), we obtain

$$\begin{aligned} |F_{\mathrm{U}} \cap (A \times B \times \ C)| &= \sum_{i,j,k} \mathbf{1}_F(i,j,k)\mathbf{1}_A(i,j)\mathbf{1}_B(j,k)\mathbf{1}_C(i,k) \\ &\leq 2s^{\frac{2a-3}{a-1}} + 2Ks^{\frac{2a-3}{a-1}}. \end{aligned}$$

Therefore, $\Psi_{F_{\mathrm{U}}}(s) \leq (2+2K)s^{\frac{2a-3}{a-1}}$, which implies

$$\dim_{\mathrm{U}} F \ \leq \ 2 - \frac{1}{a-1},$$

and hence the right-side inequality in (18).

(ii) We can assume $F = E \times \mathbb{N}$ with $\dim E > 1$, and proceed to verify $\dim_{\mathrm{U}} F \geq 2 - 1/(\dim F - 1)$. Fix $1 < a < \dim E$. Then for arbitrarily large positive integers s, there exist s-sets $A_1 \subset \mathbb{N}$, $A_2 \subset \mathbb{N}$, such that

$$|E \cap (A_1 \times A_2)| = Ms^a,$$

and $M > 0$ is as large as we please. Let $A = E \cap (A_1 \times A_2)$, $\quad B = A_1 \times [m]$ and $C = A_2 \times [m]$, where m is the integer satisfying $m - 1 < s^{a-1} \leq m$. Then,

$$|F_{\mathrm{U}} \cap (A \times B \times C)| = |A|m \approx Ms^{2a-1}.$$

Therefore, $\psi_{F_{\mathrm{U}}}(Ms^a) \geq Ms^{2a-1}$, which implies $d_{F_{\mathrm{U}}}(2 - \frac{1}{a}) = \infty$, and hence the desired conclusion.

(iii) For $v = 1, 2$, we assume

$$\sup_{k \in \mathbb{N}} d_{F_v(k)}(a-1) \leq K < \infty \tag{23}$$

(F cylindrical in directions 1 and 2), and proceed to show

$$\dim_{\mathrm{U}} F \leq \frac{\dim F}{2}. \tag{24}$$

Fix $a > \dim F$, and let A, B, C be arbitrary s-subsets of $\mathbb{N}^2$. We decompose A into two disjoint sets G and H, such that $\max_i |G_{1,i}| \leq \sqrt{s}$ and $\max_i |H_{2,i}| \leq \sqrt{s}$ (e.g., $G = \bigcup_i \{A_{2,i} : |A_{2,i}| > \sqrt{s}\}$, $\quad H = \bigcup_i \{A_{2,i} : |A_{2,i}| \leq \sqrt{s}\}$). We write

$$|F_{\mathrm{U}} \cap (A \times B \times C)| = |F_{\mathrm{U}} \cap (H \times B \times C)| + |F_{\mathrm{U}} \cap (G \times B \times C)|, \tag{25}$$

and rewrite the first term on the right side of (25) as

$$\begin{aligned} |F_{\mathrm{U}} \cap (H \times B \times C)| &= \sum_{i,j,k} \mathbf{1}_H(i,j))\mathbf{1}_B(j,k)\mathbf{1}_C(i,k)\mathbf{1}_F(i,j,k) \\ &= \sum_i \sum_{j,k} \mathbf{1}_{H_{2,i}}(j)\mathbf{1}_{B_{2,i}}(k)\mathbf{1}_C(j,k)\mathbf{1}_F(i,j,k). \end{aligned} \tag{26}$$

Let $D = \{i : |B_{2,i}| \le |H_{2,i}|\}$. By applying (23) in the case $v = 1$, we estimate

$$\begin{aligned}\sum_{i\in D}\sum_{j,k} \mathbf{1}_{H_{2,i}}(j)\mathbf{1}_{B_{2,i}}(k)\mathbf{1}_C(j,k)\mathbf{1}_F(i,j,k) &\le \sum_{i\in D}\sum_{j,k} \mathbf{1}_{H_{2,i}}(j)\mathbf{1}_{B_{2,i}}(k)\mathbf{1}_F(i,j,k)\\ &\le K\sum_{i\in D}|H_{2,i}|^{a-1}. \qquad (27)\end{aligned}$$

For $i \notin D$, let m_i be the largest integer such that

$$m_i \le \frac{|B_{2,i}|}{|H_{2,i}|} + 1, \qquad (28)$$

and decompose $B_{2,i}$ into pairwise disjoint sets $E_1, \dots, E_{m_i}$ such that $|E_u| \le |H_{2,i}|$ for $u \in [m_i]$. By applying (23) and (28), we estimate

$$\begin{aligned}\sum_{i\notin D}\sum_{j,k} \mathbf{1}_{H_{2,i}}(j)\mathbf{1}_{B_{2,i}}(k)\mathbf{1}_C(j,k)\mathbf{1}_F(i,j,k) &\le \sum_{i\notin D}\sum_{u=1}^{m_i}\sum_{j,k} \mathbf{1}_{H_{2,i}}(j)\mathbf{1}_{E_u}(k)\mathbf{1}_F(i,j,k)\\ &\le K\sum_{i\notin D}|B_{2,i}||H_{2,i}|^{a-2} + |H_{2,i}|^{a-1}.\end{aligned}$$

Combining with (27), and then using $|H_{2,i}| \le \sqrt{s}$, $\sum_i |B_{2,i}| = s$, $\sum_i |H_{2,i}| \le s$, we conclude that

$$\begin{aligned}|F_{\mathrm{U}} \cap (H \times B \times C)| &\le Ks^{\frac{a-2}{2}}\left(\sum_{i\in D}|H_{2,i}| + \sum_{i\in D^c}|B_{2,i}| + |H_{2,i}|\right)\\ &\le 2Ks^{\frac{a}{2}}.\end{aligned}$$

By a similar argument (based on (23) in the case $v = 2$), we obtain an identical estimate for the second term on the right side of (25). Combining the two estimates, we deduce

$$|F_{\mathrm{U}} \cap (A \times B \times C)| \le 4Ks^{\frac{a}{2}}.$$

Therefore, $d_{F_{\mathrm{U}}}(\frac{a}{2}) \le 4K$, which implies (24). □

4 Random constructions

Next we produce random sets (cf. [6, 7]), demonstrating that the dim-scale and the $\dim_{\mathrm{U}}$-scale implied by Theorem 7 are continuous, and are independently calibrated:

Theorem 9. **(i)** *For all* $x \in [2,3]$ *and* $y \in [\frac{x}{2}, \frac{2x-3}{x-1}]$*, there exist cylindrical sets* $F \subset \mathbb{N}^3$ *with* $\dim F = x$ *and* $\dim_{\mathrm{U}} F = y$.
(ii) *For all* $x \in [2,3]$*, there exist doubly cylindrical sets* $F \in \mathbb{N}^3$ *with* $\dim F = x$ *and (hence)* $\dim_{\mathrm{U}} F = \frac{x}{2}$.

The proof uses random constructions based on the following instance of the Prokhorov-Bennett probabilistic inequalities (e.g., [1]):

Lemma 10. *If* $(X_i : i \in \mathbb{N})$ *is a sequence of (statistically) independent* $\{0,1\}$*-valued random variables with mean* δ*, then for all* $n \in \mathbb{N}$*,*

$$\mathbb{P}\left(\Big|\sum_{i=1}^{n} X_i - n\delta\Big| > t\right) \leq 2\exp\Big(\frac{-t^2}{8n\delta}\Big), \quad t \in (0, n\delta), \tag{29}$$

$$\mathbb{P}\left(\Big|\sum_{i=1}^{n} X_i - n\delta\Big| > t\right) \leq 2\exp\Big(-\frac{t}{4}\log\big(\frac{t}{n\delta}\big)\Big), \quad t \geq 2n\delta. \tag{30}$$

In what follows below we use the notation $|A| \overset{K}{\sim} m$ to mean $m/K \leq |A| \leq Km$.

Lemma 11. *For every* $\alpha \in (1,2)$ *there is* $n_0(\alpha) = n_0 > 0$*, so that for every* $N \geq n_0$ *there exists* $E \subset [N]^2$ *such that for* $u = 1, 2$*,*

$$|E_{u,i}| \overset{2}{\sim} N^{\alpha-1} \text{ for all } 1 \leq i \leq N, \tag{31}$$

and for all s*-sets* $A \subset [N], B \subset [N]$*,*

$$|E \cap (A \times B)| \leq 5s^{\alpha}. \tag{32}$$

Proof. Fix $N \geq 1$, and let $\{X_{ij} : (i,j) \in [N]^2\}$ be a system of independent $\{0,1\}$-value random variables with mean $N^{\alpha-2}$. Consider the random set $E = \{(i,j) : X_{ij} = 1\}$. The probability that E fails to satisfy (31) is no larger than

$$\sum_{i=1}^{N}\mathbb{P}\left(\left|\sum_{i=1}^{N} X_{ij} - N^{\alpha-1}\right| > N^{\alpha-1}/2\right) + \sum_{j=1}^{N}\mathbb{P}\left(\left|\sum_{j=1}^{N} X_{ij} - N^{\alpha-1}\right| > N^{\alpha-1}/2\right),$$

which, following (29), is no larger than $1/3$ for all $N \geq n_0$ for sufficiently large n_0. Also, the probability that E fails to satisfy (32) is no larger than

$$\sum_{s=1}^{N}\sum_{|A|=s,|B|=s}\mathbb{P}\left(\sum_{(i,j)\in A\times B} X_{ij} - s^2N^{\alpha-2} > 4s^{\alpha}\right),$$

which, following (30), is no larger than

$$\sum_{s=1}^{N}\binom{N}{s}^2 \cdot 2\exp\left(-(2-\alpha)s^{\alpha}\log\left(\frac{2N}{s}\right)\right) < \frac{1}{3}$$

for all $N \geq n_0$ for sufficiently large n_0. We thus conclude that E satisfies the requirements of the lemma with probability at least $1/3$. □

Lemma 12. *For every* $\alpha \in (1,2)$ *and every* $\beta \in [0, \alpha-1)$ *there is* $n_0(\alpha,\beta) = n_0 > 0$*, so that for every* $N \geq n_0$ *there exists* $F \subset [N]^3$ *satisfying the following:*
(i) *there exists* $E \subset [N]^2$ *with* $|E| \overset{2}{\sim} N^{\alpha}$*, such that for all* $k \in [N]$*,*

$$|\{(i,j) : (i,j,k) \in F,\ (i,j) \in E\}| \overset{4}{\sim} N^{\alpha-\beta}; \tag{33}$$

(**ii**) *for all* s*-sets* $A \subset [N]$ *and* $B \subset [N]$*, and for all* $k \in [N]$*,*

$$|F \cap (A \times B \times \{k\})| \leq 25 s^{\alpha-\beta}; \tag{34}$$

(**iii**) *for all* s*-sets* $S \subset [N] \times [N]$*, and for all* $C \subset [N]$ *with* $|C| \leq s^{1-1/\alpha}$*,*

$$|F \cap (S \times C)| \leq 5 s^{2-1/\alpha-\beta/\alpha}. \tag{35}$$

Proof. Choose E as in Lemma 11. If $\beta = 0$, then we simply let $F = E \times [N]$. For $\beta \in (0, \alpha-1)$, let $\{X_{ijk} : (i,j) \in E, 1 \leq k \leq N\}$ be a system of independent $\{0,1\}$-valued random variables with mean $N^{-\beta}$. Then, by argument similar to the one used to prove Lemma 11, we conclude that $F = \{(i,j,k) : X_{ijk} = 1\}$ satisfies the requirements of the lemma with positive probability. □

Proof of Theorem 9. (**i**) For $x = 2$, take $F = \mathbb{N}^2 \times \{1\}$, and for $x = 3$, take $F = \mathbb{N}^3$. For $x \in (2,3)$, let $\alpha = \dfrac{x-2}{y-1}$, and $\beta = \alpha + 1 - x$. Then, $1 < \alpha \leq 2$, and $0 \leq \beta < \alpha - 1$. Let $n_0 = n_0(\alpha, \beta)$ be as in Lemma 11, and for each integer n such that $3^n \geq n_0$, let F_n be the set F obtained from Lemma 11 for $N = 3^n$. Define

$$F = \bigcup_{n=n_0}^{\infty} \{(3^n, 3^n, 3^n) + F_n\}.$$

Claim 1: $\dim F = x$.

For all s-sets $A \subset \mathbb{N}$, $B \subset \mathbb{N}$, $C \subset \mathbb{N}$,

$$\begin{aligned} |F \cap (A \times B \times C)| &= \sum_{n=n_0}^{\infty} \left| \big((3^n, 3^n, 3^n) + F_n\big) \cap (A \times B \times C) \right| \\ &= \sum_{n=n_0}^{\infty} \left| F_n \cap \big((A - 3^n) \times (B - 3^n) \times (C - 3^n)\big) \right| \\ &\leq \sum_{n=n_0}^{\infty} 25 s_n^x, \end{aligned}$$

where

$$s_n = \max\{|(A - 3^n) \cap [3^n]|, |(B - 3^n) \cap [3^n]|, |(C - 3^n) \cap [3^n]|\},$$

and the inequality follows from (34).

Note that

$$\begin{aligned} \sum_{n=n_0}^{\infty} s_n^x &\leq \left(\sum_{n=n_0}^{\infty} s_n \right)^x \\ &\leq \left(\sum_{n=n_0}^{\infty} (|A \cap (3^n + [3^n])| + |B \cap (3^n + [3^n])| + |C \cap (3^n + [3^n])|) \right)^x \\ &\leq (|A| + |B| + |C|)^x = 3^x s^x. \end{aligned}$$

We thus conclude

$$\Psi_F(s) \le 25 \cdot 3^x s^x. \tag{36}$$

On the other hand, choosing $A = B = C = 3^n + [3^n]$, we obtain from (33)

$$|F \cap (A \times B \times C)| = |(3^n, 3^n, 3^n) + F_n| = |F_n| \ge \frac{1}{4}(3^n)^{\alpha-\beta+1} = \frac{1}{4}(3^n)^x.$$

Therefore $\Psi_F(3^n) \ge \frac{1}{4}(3^n)^x$, which, together with (36), implies $\dim F = x$.

Claim 2: $\dim_U F = y$.

We let $A \subset \mathbb{N}^2$, $B \subset \mathbb{N}^2$, $C \subset \mathbb{N}^2$ be s-sets, and estimate $|F_{\mathrm{U}} \cap (A \times B \times C)|$. To this end, let

$$H = \{k : |B_{1,k}| > s^{1/\alpha} \text{ or } |C_2, k| > s^{1/\alpha}\}.$$

Then, $|H| \le 2s^{1-1/\alpha}$ (because $\sum(|B_{1,k}| + |C_{2,k}|) = |B| + |C| = 2s$). We note that $|F_{\mathrm{U}} \cap (A \times B \times C)|$ is less than or equal to

$$\sum_{k \notin H} |F \cap (C_{2,k} \times B_{1,k} \times \{k\})| + \sum_{k \in H} |F \cap (A \times \{k\})|. \tag{37}$$

To estimate the first sum in (37), we observe that for every $k \in \mathbb{N}$,

$$F \cap (C_{2,k} \times B_{1,k} \times \{k\}) = ((3^l, 3^l, 3^l) + F_l) \cap (C_{2,k} \times B_{1,k} \times \{k\}),$$

where $3^l \le k < 3^{l+1}$. By (34),

$$\begin{aligned} & \left|((3^l, 3^l, 3^l) + F_l) \cap (C_{2,k} \times B_{1,k} \times \{k\})\right| \\ = & \left|F_l \cap ((C_{2,k} - 3^l) \times (B_{1,k} - 3^l) \times (\{k\} - 3^l))\right| \\ \le & \ 25\,(\max\{|B_{1,k}|, |C_{2,k}|\})^{\alpha-\beta}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k \notin H} |F \cap (C_{2,k} \times B_{1,k} \times \{k\})| &\le \sum_{k \notin H} 25\,(\max\{|B_{1,k}|, |C_{2,k}|\})^{\alpha-\beta} \\ &\le 25 \sum_{k \notin H} (\max\{|B_{1,k}|, |C_{2,k}|\})\,(s^{1/\alpha})^{\alpha-\beta-1} \\ &\le 25 \cdot 2s \cdot s^{1-(1+\beta)/\alpha} \\ &= 50 s^{2-(1+\beta)/\alpha}. \end{aligned} \tag{38}$$

To estimate the second sum in (37), denote

$$A^{(l)} = A \cap ((3^l, 3^l) + [3^l] \times [3^l]), \qquad H^{(l)} = H \cap (3^l + [3^l]).$$

Then

$$\begin{aligned}\sum_{k\in H}|F\cap(A\times\{k\})| &= \sum_{l=1}^{\infty}|(3^l,3^l,3^l)+F_l\cap(A^{(l)}\times H^{(l)})| \\ &= \sum_{l=1}^{\infty}|F_l\cap((A^{(l)}-(3^l,3^l))\times(H^{(l)}-3^l))|. \qquad (39)\end{aligned}$$

By (35), each summand in (39) is bounded by

$$\begin{cases} 5|A^{(l)}|^{2-\frac{1}{\alpha}-\frac{\beta}{\alpha}} & \text{if } |H^{(l)}|\le|A^{(l)}|^{1-\frac{1}{\alpha}}, \\ 5|H^{(l)}|^{(2-\frac{1}{\alpha}-\frac{\beta}{\alpha})/(1-\frac{1}{\alpha})} & \text{if } |H^{(l)}|>|A^{(l)}|^{1-\frac{1}{\alpha}}. \end{cases}$$

Therefore,

$$\begin{aligned}\sum_{k\in H}|F\cap(A\times\{k\})| &\le \sum_{l=1}^{\infty}5|A^{(l)}|^{2-\frac{1}{\alpha}-\frac{\beta}{\alpha}}+\sum_{l=1}^{\infty}5|H^{(l)}|^{(2-\frac{1}{\alpha}-\frac{\beta}{\alpha})/(1-\frac{1}{\alpha})} \\ &\le 5\left(\sum_{l=1}^{\infty}|A^{(l)}|\right)^{2-\frac{1}{\alpha}-\frac{\beta}{\alpha}}+5\left(\sum_{l=1}^{\infty}|H^{(l)}|\right)^{(2-\frac{1}{\alpha}-\frac{\beta}{\alpha})/(1-\frac{1}{\alpha})} \\ &\le 5|A|^{2-\frac{1}{\alpha}-\frac{\beta}{\alpha}}+5|H|^{(2-\frac{1}{\alpha}-\frac{\beta}{\alpha})/(1-\frac{1}{\alpha})} \\ &\le 5(1+2^{\frac{\alpha}{\alpha-1}})s^{2-\frac{1}{\alpha}-\frac{\beta}{\alpha}}, \qquad (40)\end{aligned}$$

where the last inequality holds because $|H|\le 2s^{1-1/\alpha}$. By combining (37), (38) and (40), we conclude that

$$\Psi_{F_{\mathrm{U}}}(s)\le\big(50+5(1+2^{\alpha/(\alpha-1)})\big)s^{2-1/\alpha-\beta/\alpha}=5(11+2^{\alpha/(\alpha-1)})s^y, \qquad (41)$$

which implies $\dim_{\mathrm{U}}F\le y$.

To obtain the opposite inequality, for $n>n_0$, let m be the integer such that $m-1<(3^n)^{\alpha-1}\le m$. Let $A=(3^n,3^n)+E_n$, where $E_n=E$ is obtained from Lemma 12(i) for $N=3^n$, and let $B=(3^n,3^n)+[3^n]\times[m]$ and $C=(3^n,3^n)+[m]\times[3^n]$. Then $|A|\overset{2}{\sim}3^{\alpha n}$, and $|B|=|C|<3^{\alpha n}+3^n$. Applying (33), we obtain

$$\begin{aligned}|F_{\mathrm{U}}\cap(A\times B\times C)| &= |\{(i,j,k)\in(3^n,3^n,3^n)+F_n:(i,j)\in A,(j,k)\in B,(k,i)\in C\}| \\ &= |\{(i,j,k)\in F_n:(i,j)\in E_n,\ k\in[m]\}| \\ &\overset{4}{\sim} (3^n)^{\alpha-\beta}\cdot m \\ &\ge \frac{1}{4}(3^{\alpha n})^y.\end{aligned}$$

Therefore, $\Psi_{F_{\mathrm{U}}}(2\cdot 3^{\alpha n})\ge\frac{1}{4}(3^{\alpha n})^y$, which implies $\dim_{\mathrm{U}}F\ge y$.

Claim 3: F is cylindrical.

We have shown that $\dim F=\alpha-\beta+1$. For every $k\in\mathbb{N}$, $\pi_3^{-1}\{k\}\cap F\ne\emptyset$ only if $3^l\le k\le 2\cdot 3^l$, in which case

$$\pi_3^{-1}\{k\}\cap F=\pi_3^{-1}\{k\}\cap((3^l,3^l,3^l)+F_l)=(3^l,3^l,3^l)+\pi_3^{-1}\{k-3^l\}\cap F_l.$$

By (34), $d_{\pi_3^{-1}\{k\}\cap F_{3^l}}(\alpha-\beta) \leq 5$ for all $1 \leq k \leq 3^l$. Therefore, $d_{\pi_3^{-1}\{k\}\cap F}(\alpha-\beta) \leq 5$ for all $k \in \mathbb{N}$, and hence F is cylindrical in direction 3.

(ii) The construction of (random) doubly-cylindrical sets with the desired dimension follows a blueprint similar to the one followed in **(i)**. Indeed, if $x \in (2, 3)$, then for all $N \geq n_0$ for sufficiently large n_0, by using independent $\{0, 1\}$-valued random variables with a prescribed mean, we can produce (random) sets $F_N \subset [N]^3$ with the following properties: (a) for every positive integer $s \leq N$, s-subsets A and B of $[N]$, and every $l \in [N]$,

$$|F_N \cap (A \times B \times \{l\}| \leq s^{x-1}, \ |F_N \cap (A \times \{l\} \times B| \leq s^{x-1}, \ |F_N \cap (\{l\} \times A \times B| \leq s^{x-1};$$

(b) for every $l \in [N]$,

$$|F_N \cap ([N]^2 \times \{l\}| \overset{2}{\sim} N^{x-1}, \ |F_N \cap ([N] \times \{l\} \times [N]| \overset{2}{\sim} N^{x-1}, \ |F_N \cap (\{l\} \times [N]^2| \overset{2}{\sim} N^{x-1}.$$

Then, $F = \cup_{n=n_0}^{\infty}(F_{3^n} + (3^n, 3^n, 3^n))$ is cylindrical in each of the three directions, and $\dim F = x/2$. The verification is similar to the proof given in **(i)**, and is omitted. □

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