A Characterization of Random Bloch Functions

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Abstract
In this paper, we introduce a necessary and sufficient condition on the complex sequence \( \{a_n\} \), \( \sum |a_n|^2 < \infty \), so that \( \sum_{n=1}^{\infty} \pm a_n z^n \) represents a Bloch function for almost all choices of signs “\( \pm \)”, answering a question left open in [2].

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Introduction

A Bloch function is an analytic function \( f(z) \) in the unit disk \( D = \{ z : |z| < 1 \} \), such that
\[
\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.
\]
When equipped with the norm
\[
\| f \|_B = |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)|,
\]
the set of all Bloch functions forms a Banach space, called the Bloch space.
In this note, we study the random power series
\[
f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n
\]
where \( \{\varepsilon_n(\omega)\} \) is a Rademacher sequence, that is \( \varepsilon_n = \pm 1 \). In particular, we will consider the following problem raised by Anderson in [1]:

Problem Find a necessary and sufficient condition on \( \{a_n\} \), such that for Rademacher sequence \( \{\varepsilon_n(\omega)\} \), the series
\[
f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n
\]
represents a Bloch function almost surely.
For the history and the related research, see e.g. [2], [3] and [1].
The study of random series dates back at least to Paley and Zygmund (1930). For a long time, a major question was to characterize the a.s. convergence of the random Fourier series
\[ \sum_{n=0}^{\infty} a_n \varepsilon_n e^{ni\theta}, \]
where \( \{a_n\} \) is a sequence of numbers satisfying \( \sum_{n=0}^{\infty} |a_n|^2 < \infty \). This question was completely solved by Marcus and Pisier ([4]). Their result will be adapted in this paper to produce the proof of the sufficient part of the following theorem.

**Theorem 1** If \( \{\varepsilon_n\} \) is a Rademacher sequence, then the random power series
\[ f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n \]
is a Bloch function almost surely if and only if
\[ \int_0^{\infty} \overline{d_n}(e^{-t^2}) dt = O(n), \]
where \( \overline{d_n} \) is the non-decreasing rearrangement of
\[ d_n(t) = \left( \sum_{k=1}^{n} k^2 |a_k|^2 |e^{2\pi k t} - 1|^2 \right)^{1/2}. \]

Here and throughout this note, the non-decreasing rearrangement of a (Lebesgue) \( m \)-measurable function \( h(t) \) on \([0, 1]\) is defined by
\[ \overline{h}(s) = \sup \{ y : m(\{ t : h(t) < y \}) < s \}. \]

**Marcus-Pisier**

In this section, we introduce a result of Marcus and Pisier [4]. For the notational simplicity, we define \( \overline{\rho}(t) \) to be the non-decreasing rearrangement of
\[ \rho(t) = \left( \sum_{n=0}^{\infty} |a_n|^2 |e^{2\pi n t} - 1|^2 \right)^{1/2}, \]
and denote
\[ I := \int_0^1 \frac{\overline{\rho}(t)}{t \sqrt{-\log t}} dt. \]
The following result can be found in [4] (p.11, Th. 1.4).
Proposition 1 (Marcus-Pisier) Let \( \{\xi_n\} \) be a sequence of independent, symmetric random variables. Then there exists a constant \( K \), such that

\[
\frac{1}{K} \left( \inf_n E|\xi_n| \right) \left[ \sum_{n=0}^{\infty} |a_n|^2 + 1 \right] \leq EZ \leq K \sqrt{\sup_n E|\xi_n|^2} \left[ \sum_{n=0}^{\infty} |a_n|^2 + 1 \right]
\]

where

\[
Z := \sup_{0 \leq \theta < 2\pi} \left| \sum_{n=0}^{\infty} a_n e^{n\theta i} \xi_n(\omega) \right|.
\]

For our purpose, we need to improve the right inequality to the following

Proposition 2 There exists a constant \( C \), such that

\[
\left\| \sup_{0 \leq \theta < 2\pi} \sum_{n=0}^{\infty} a_n e^{n\theta i} \xi_n(\omega) \right\|_{\psi_2} \leq C \left[ \sqrt{\sum_{n=0}^{\infty} |a_n|^2 + 1} \right],
\]

where the Orlicz norm \( \| \cdot \|_{\psi_2} \) is defined by the equation

\[
\|x\|_{\psi_2} := \inf \left\{ c > 0 : E \exp \left( \frac{|x|^2}{c^2} \right) = 2 \right\}.
\]

To prove Proposition 2, we need two lemmas. Lemma 1 ([6], p. 43, Theorem 2.1) is called Maurey-Pisier concentration inequality; Lemma 2 ([5], p.97) is a consequence of the contraction principle (see Lemma 3 in the next section).

Lemma 1 Let \( \{X_t\}_{t \in T} \) be a centered Gaussian processes with sample paths bounded a.s. Let \( \sigma := \sup_{t \in T} EX_t^2 \). Then

\[
P\left\{ \sup_{t \in T} X_t - E \sup_{t \in T} X_t > \lambda \right\} \leq 2 \exp \left\{ -\frac{\lambda^2}{2\sigma^2} \right\}.
\]

Lemma 2 If \( \{g_i(\omega)\} \) is a sequence of i.i.d standard normal random variables, then

\[
\left\| \sup_{0 \leq \theta < 2\pi} \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega) \right\|_{\psi_2} \leq \sqrt{\frac{\pi}{2}} \sup_{0 \leq \theta < 2\pi} \left\| \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega) \right\|_{\psi_2}.
\]

Proof: Let \( \{g_i(\omega)\} \) be a sequence of i.i.d standard normal random variables. Denote

\[
Y_g := \sqrt{\frac{\pi}{2}} \sup_{0 \leq \theta < 2\pi} \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega)
\]

and

\[
Z_g := \sqrt{\frac{\pi}{2}} \sup_{0 \leq \theta < 2\pi} \left| \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega) \right|,
\]
By the symmetry of Gaussian variables, we have

\[ P\{Z_g > \lambda\} \leq 2P\{Y_g > \lambda\}. \]

Using this inequality and then applying Lemma 1 to \(Y_g\), we obtain

\[ \|Z_g\|_{\psi_2} \leq 2\|Y_g\|_{\psi_2} \leq C \left( EY_g + \sqrt{\sum_{n=0}^{\infty} |a_n|^2} \right) \]

for some constant \(C\). On the other hand, by applying Proposition 1 to \(Z_g\), we have

\[ EZ_g \leq K \left[ \sum_{n=0}^{\infty} |a_k|^2 + I \right] \]

for some constant \(K\). The Proposition follows by invoking Lemma 2.

\[ \square \]

**Proof of Theorem 1**

We will need the following contraction principle ([5], p. 95, Theorem 4.4).

**Lemma 3** Let \(F: \mathbb{R}_+ \to \mathbb{R}_+\) be convex. For any finite sequence \((x_k)\) in a Banach space \(B\) and any real numbers \((\alpha_k)\) such that \(|\alpha_k| \leq 1\) for every \(k\), we have

\[ EF\left(\left\| \sum_k \alpha_k \varepsilon_k x_k \right\|\right) \leq EF\left(\left\| \sum_k \varepsilon_k x_k \right\|\right). \]

We start with the following identity. For \(z = re^{i\theta}\),

\[ (1 - |z|) |f'_\omega(z)| = (1 - |z|) \left| \sum_{n=1}^{\infty} na_n z^{n-1} \varepsilon_n \right| \]

\[ = (1 - r) \left| \sum_{n=1}^{\infty} nr^{n-1} a_n e^{ni\theta} \varepsilon_n \right| \]

\[ = \left| \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} k a_k e^{ki\theta} \varepsilon_k \right) r^{n-1} (1 - r)^2 \right|. \]

(i) Suppose

\[ \int_0^\infty e^{-t^2} dt = O(n). \]

By changing variable, this is equivalent to

\[ \int_0^1 \frac{d_n(t)}{t \sqrt{-\log t}} dt = O(n). \]
Applying Proposition 2 to the random series \( \sum_{k=0}^{n} k a_k e^{k \theta_i} \varepsilon_k \), we have
\[
\left\| \sup_{0 \leq \theta < 2\pi} \left| \sum_{k=0}^{n} k a_k e^{k \theta_i} \varepsilon_k \right| \right\|_{\psi_2} \leq C \left[ \sqrt{\sum_{k=0}^{n} |a_k|^2} + \int_{0}^{\infty} \frac{\overline{d_n}(t)}{t^{\sqrt{-\log t}}} dt \right].
\]

By Chebyshev’s inequality, we deduce that
\[
\sup_{0 \leq \theta < 2\pi} \left| \sum_{k=1}^{n} k a_k e^{k \theta_i} \varepsilon_k \right| \leq n + C\left( \sqrt{\sum_{k=1}^{n} |a_k|^2} + \int_{0}^{\infty} \frac{\overline{d_n}(t)}{t^{\sqrt{-\log t}}} dt \right)
\]
\[
\leq n + C n \left( \sum_{k=1}^{n} |a_k|^2 + \int_{0}^{\infty} \frac{\overline{d_n}(t)}{t^{\sqrt{-\log t}}} dt \right)
\]
\[
\leq C'n
\]
except on a set with probability less than \( e^{-n} \). (The purpose of Proposition 2 is to produce this quantity.) Thus, with probability more than \( 1 - \sum_{n=m}^{\infty} e^{-n} \), we have
\[
\sup_{z \in D} (1 - |z|) |f_{\omega}'(z)| = \sup_{0 < r < 1} \sup_{0 \leq \theta \leq 2\pi} \left| \sum_{n=1}^{\infty} r^{n-1} (1 - r)^2 \sum_{k=1}^{n} k a_k e^{k \theta_i} \varepsilon_k \right|
\]
\[
\leq C_m + \sup_{0 < r < 1} \sum_{n=m}^{\infty} r^{n-1} (1 - r)^2 \sup_{0 \leq \theta \leq 2\pi} \left| \sum_{k=1}^{n} k a_k e^{k \theta_i} \varepsilon_k \right|
\]
\[
\leq C_m + \sum_{n=m}^{\infty} r^{n-1} (1 - r)^2 C'n
\]
\[
\leq C_m + C' < \infty
\]
where \( C_m \) is a constant depending on \( m \). This implies \( f_{\omega}(z) \) is a Bloch function almost surely.

(ii) Suppose \( f_{\omega}(z) \) is a Bloch function almost surely. Then
\[
\sup_{z \in D} (1 - |z|) |f_{\omega}'(z)| < \infty.
\]

By changing variable, and applying the left inequality of Proposition 1 to the series \( \sum_{k=1}^{n} k a_k e^{k \theta_i} \varepsilon_k(\omega) \), we have
\[
\int_{0}^{\infty} \overline{d_n}(e^{-t^2}) dt = 2 \int_{0}^{1} \frac{\overline{d_n}(t)}{t^{\sqrt{-\log t}}} dt
\]
\[
\leq 2KE \sup_{\theta} \left| \sum_{k=1}^{n} k a_k e^{k \theta_i} \varepsilon_k(\omega) \right|.
\]
Consider
\[
\frac{1}{n} E \sup_{\theta} \left| \sum_{k=1}^{n} k a_k e^{k \theta_i} \varepsilon_k(\omega) \right|.
\]
Because for $k \leq n$, \( \left( 1 - \frac{1}{n} \right)^k \geq \frac{1}{e} \), by the contraction principle (Lemma 3),

\[
\frac{1}{n} E \sup_{\theta} \left| \sum_{k=1}^{n} k a_k e^{k i \theta} \varepsilon_k(\omega) \right| \leq e E \sup_{\theta} \left| \sum_{k=1}^{n} k a_k e^{k i \theta} \left( 1 - \frac{1}{n} \right)^k \varepsilon_k(\omega) \right|
\leq e E \sup_{\theta} \left| \sum_{k=1}^{\infty} k a_k e^{k i \theta} \left( 1 - \frac{1}{n} \right)^k \varepsilon_k(\omega) \right|
\leq e E \sup_{\theta} \left| \sum_{k=1}^{\infty} k a_k (1 - r^k) e^{k i \theta} \varepsilon_k(\omega) \right|
= e E \sup_{\theta} (1 - |z|) \left| \sum_{k=1}^{\infty} k a_k z^k \varepsilon_k(\omega) \right|
< \infty,
\]

which implies that

\[
\int_{0}^{\infty} d_n(e^{-t^2})dt = O(n).
\]

**Corollary 1 (see [2])** If

\[
\sqrt{\sum_{k=1}^{n} |a_k|^2 k^2} = O\left( \frac{n}{\sqrt{\log n}} \right),
\]

then $\sum_{n=0}^{\infty} a_n e_n z^n$ represents a Bloch function almost surely.

**Proof:**

\[
\int_{0}^{\infty} d_n(e^{-t^2})dt \leq \int_{0}^{\infty} \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2} \exp(2\pi k e^{-t^2 i}) - 1^2 dt
\leq 2 \int_{0}^{\sqrt{\log n}} \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2} dt + 8\pi^2 \int_{\sqrt{\log n}}^{\infty} \sqrt{\sum_{k=1}^{n} k^4 |a_k|^2 e^{-t^2}} dt
\leq 2 \sqrt{\log n} \cdot \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2} + 8\pi^2 \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2}
= O(n).
\]

The Corollary then follows from Theorem 1.

**Remark:** (i) The readers who are familiar with Marcus-Pisier’s proof of Proposition 1 (the idea of replacing a symmetric random variable $\xi_n$ by an identically distributed random variable $\xi_n \epsilon_n$) should have noticed that Theorem 1 remains valid if $\epsilon_n$’s are replaced by the $\xi_n$’s in Proposition 1. (ii) Anderson also asked the question of characterizing
random BMO functions, to which Duren had a very sharp sufficient condition. We note
that Duren’s sufficient condition can be replaced by a sharper Maurey-Pisier type con-
dition. However, the technique that we used in this paper seems not to work in finding
the necessary condition.

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