INTRINSIC VOLUMES AND POLAR SETS IN SPHERICAL SPACE

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Abstract. For a convex body of given volume in spherical space, the total invariant measure of hitting great subspheres becomes minimal, equivalently the volume of the polar body becomes maximal, if and only if the body is a spherical cap. This result can be considered as a spherical counterpart of two Euclidean inequalities, the Urysohn inequality connecting mean width and volume, and the Blaschke-Santaló inequality for the volumes of polar convex bodies. Two proofs are given; the first one can be adapted to hyperbolic space.

MSC 2000: 52A40 (primary); 52A55, 52A22 (secondary)

Keywords: spherical convexity, Urysohn inequality, Blaschke-Santaló inequality, intrinsic volume, quermassintegral, polar convex bodies, two-point symmetrization

In the classical geometry of convex bodies in Euclidean space, the intrinsic volumes (or quermassintegrals) play a central role. These functionals have counterparts in spherical and hyperbolic geometry, and it is a challenge for a geometer to find analogues in these spaces for the known results about Euclidean intrinsic volumes. In the present paper, we prove one such result, an inequality of isoperimetric type in spherical (or hyperbolic) space, which can be considered as a counterpart to the Urysohn inequality and is, in the spherical case, also related to the Blaschke-Santaló inequality. Before stating the result (at the beginning of Section 2), we want to recall and collect the basic facts about intrinsic volumes in Euclidean space, describe their noneuclidean counterparts, and state those open problems about the latter which we consider to be of particular interest. We restrict ourselves here to spherical space and comment only briefly on the case of hyperbolic space.

1 Intrinsic volumes

By $\mathbb{E}^n$ we denote the n-dimensional Euclidean vector space, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Its unit ball and unit sphere are $B^n := \{x \in$

$\mathbb{E}^n : \|x\| \leq 1\}$ and $\mathbb{S}^{n-1} := \{x \in \mathbb{E}^n : \|x\| = 1\}$, respectively. Lebesgue measure on $\mathbb{E}^n$ is denoted by λ_n, and spherical Lebesgue measure on $\mathbb{S}^{n-1}$ by σ_{n-1}. Then $\kappa_n := \lambda_n(B^n) = \pi^{n/2}/\Gamma(1+n/2)$ and $\beta_{n-1} := \sigma_{n-1}(\mathbb{S}^{n-1}) = n\kappa_n = 2\pi^{n/2}/\Gamma(n/2)$. By $\mathcal{K}^n$ we denote the space of convex bodies (nonempty, compact, convex sets) in $\mathbb{E}^n$.

For $K \in \mathcal{K}^n$ and $\epsilon > 0$, the parallel body of K at distance ϵ is given by

$$K_\epsilon := \{x \in \mathbb{E}^n : d(K, x) \leq \epsilon\},$$

where $d(K, \cdot)$ denotes the Euclidean distance from K. By Steiner's formula, there is a polynomial expansion

$$\lambda_n(K_\epsilon) = \sum_{j=0}^{n} \epsilon^{n-j} \kappa_{n-j} V_j(K) = \sum_{i=0}^{n} \epsilon^i \binom{n}{i} W_i(K) \tag{1}$$

for $\epsilon \geq 0$. This defines the jth *intrinsic volume* $V_j : \mathcal{K}^n \to \mathbb{R}$, and also the ith *quermassintegral* $W_i : \mathcal{K}^n \to \mathbb{R}$. We use here only the first normalization. Particular cases are $2V_{n-1}$, the surface area, and $(2\kappa_{n-1}/n\kappa_n)V_1$, the mean width. Special representations hold under additional assumptions. If the boundary ∂K of K is a regular C^2 hypersurface, then

$$V_j(K) = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int_{\partial K} H_{n-1-j} \, dA, \tag{2}$$

where H_k is the kth normalized elementary symmetric function of the principal curvatures of ∂K and dA denotes the area element. If P is a polytope and $\mathcal{F}_j(P)$ denotes the set of its j-dimensional faces, then

$$V_j(P) = \sum_{F \in \mathcal{F}_j(P)} \gamma(F, P) \lambda_j(F), \tag{3}$$

where $\gamma(F, P)$ is the normalized external angle of P at its face F.

The Gauss-Bonnet theorem for convex bodies says that

$$V_0(K) = \chi(K), \tag{4}$$

where χ denotes the Euler characteristic ($\chi(K) = 1$ for $K \in \mathcal{K}^n$).

The fact that V_j is, up to a factor, a special mixed volume, is not relevant in our present context, since the notion of mixed volume is based on the vector addition in $\mathbb{E}^n$ and has, therefore, no direct noneuclidean counterpart. Of relevance, however, is the integral-geometric interpretation (which gave rise, historically, to the term 'quermassintegral').

Let $\mathcal{E}^n_q$ be the space of q-dimensional planes in $\mathbb{E}^n$, and let μ_q be its motion invariant measure, normalized so that $\mu_q(\{E \in \mathcal{E}^n_q : E \cap B^n \neq \emptyset\}) = \kappa_{n-q}$. Then

$$V_j(K) = \binom{n}{j} \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \int_{\mathcal{E}^n_{n-j}} \chi(K \cap E) \, d\mu_{n-j}(E). \tag{5}$$

Since $\chi(K \cap E) = 1$ if $K \cap E \neq \emptyset$, $V_j(K)$ is, up to a factor, the total invariant measure of the set of $(n-j)$-planes meeting K.

Let $\mathcal{L}_q^n$ denote the Grassmannian of q-dimensional linear subspaces of $\mathbb{E}^n$ and let ν_q be its rotation invariant probability measure. For $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_q^n$, we denote by $K|L$ the image of K under orthogonal projection to L. Then

$$V_j(K) = \binom{n}{j} \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \int_{\mathcal{L}_j^n} \lambda_j(K|L)\, d\nu_j(L). \tag{6}$$

Thus $V_j(K)$ is, up to a factor, the average volume of the j-dimensional projections of K. By a suitable decomposition of μ_{n-j}, formula (5) is an immediate consequence of (6).

As functions on $\mathcal{K}^n$, the intrinsic volumes are invariant under rigid motions, nonnegative, increasing under set inclusion, continuous with respect to the Hausdorff metric, and additive. A function $\varphi : \mathcal{K}^n \to \mathbb{R}$ is *additive* or a *valuation* if $\varphi(K_1 \cup K_2) + \varphi(K_1 \cap K_2) = \varphi(K_1) + \varphi(K_2)$ whenever $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n$.

Hadwiger's celebrated characterization theorem says: *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is additive, rigid motion invariant and continuous, then $\varphi = \sum_{j=0}^n c_j V_j$ with constant coefficients $c_0, \ldots, c_n$.* See Hadwiger [14, 6.1.10] for the classical approach to this theorem; for a shorter proof, see Klain [16] and Klain-Rota [17, 9.1].

The intrinsic volumes satisfy various sharp inequalities, resulting from the theory of mixed volumes. We mention here only the inequalities connecting two intrinsic volumes. For $K \in \mathcal{K}^n$ and $1 \leq j < k \leq n$, one has

$$\left(\frac{\kappa_{n-j}}{\binom{n}{j}} V_j(K) \right)^k \geq \kappa_n^{k-j} \left(\frac{\kappa_{n-k}}{\binom{n}{k}} V_k(K) \right)^j \tag{7}$$

(see, e.g., [28, p. 334]). Equality holds if K is a ball, and if $V_j(K) > 0$ it holds only in this case. In particular, among all convex bodies of given positive volume, precisely the balls have the least jth intrinsic volume, for $j = 1, \ldots, n-1$.

Now we discuss the situation in spherical space. For n-dimensional spherical space, we take the unit sphere $\mathbb{S}^n$ of $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. By a convex body in $\mathbb{S}^n$ we understand here the intersection $\mathbb{S}^n \cap C$ of $\mathbb{S}^n$ with a line-free closed convex cone C, where $\{0\} \neq C \subset \mathbb{E}^{n+1}$. Thus, a convex body in this sense is contained in some open hemisphere. We denote by $\mathcal{K}_s^n$ the space of all convex bodies in $\mathbb{S}^n$, equipped with the Hausdorff metric.

For $A \subset \mathbb{S}^n$ and $\epsilon \geq 0$, the parallel set A_ϵ is defined by

$$A_\epsilon := \{x \in \mathbb{S}^n : d_s(A, x) \leq \epsilon\}, \tag{8}$$

where $d_s(A,x)$ denotes the spherical distance of x from A. For $K \in \mathcal{K}_s^n$ and $0 \le \epsilon \le \pi/2$, Steiner's formula in $\mathbb{S}^n$ can be written as

$$\sigma_n(K_\epsilon) = \beta_n V_n(K) + \sum_{j=0}^{n-1} f_j(\epsilon)\beta_j\beta_{n-j-1}V_j(K), \tag{9}$$

where $V_n(K) = \sigma_n(K)/\beta_n$ and

$$f_j(\epsilon) := \int_0^\epsilon \cos^j t \sin^{n-j-1} t \, dt.$$

Older references for Steiner formulae in spaces of constant curvature are Herglotz [15], Allendoerfer [1], Santaló [22]. A very general (local) version (for sets of positive reach) appears in Kohlmann [18]. For convex bodies in spherical space, an elementary proof, using polytopes and approximation, was given by Glasauer [10].

Since the functions $f_0, \dots, f_{n-1}$ are linearly independent, the expansion (9) defines the coefficients $V_j(K)$ uniquely. The chosen normalization is convenient. We call the functionals $V_0, \dots, V_n$ the *intrinsic volumes* in $\mathbb{S}^n$, although other functionals discussed below might also deserve this name.

As a counterpart to (2), we now have, under smoothness assumptions,

$$V_j(K) = \frac{\binom{n-1}{j}}{\beta_j\beta_{n-j-1}} \int_{\partial K} H_{n-1-j} \, dA, \tag{10}$$

with analogous meanings of H_k and dA. In addition to the literature quoted above, we refer to Santaló [24] (with different notation).

The counterpart to (3) we formulate in a slightly different way. Let $P \in \mathcal{K}_s^n$ be a spherical polytope, thus $P = \mathbb{S}^n \cap C_P$, where C_P is a polyhedral cone in $\mathbb{E}^{n+1}$. For faces F, G of C_P with $F \subset G$, we denote by $\beta(F,G)$ the internal and by $\gamma(F,G)$ the external (normalized) angle of G at F. Then

$$V_j(P) = \sum_{F \in \mathcal{F}_{j+1}(C_P)} \beta(0,F)\gamma(F,C_P), \tag{11}$$

where $\mathcal{F}_{j+1}(C_P)$ is the set of $(j+1)$-faces of the polyhedral cone C_P.

In spherical space, the intrinsic volumes behave well under duality. For $A \subset \mathbb{S}^n$, the polar set is defined by

$$A^* := \{x \in \mathbb{S}^n : \langle x, a\rangle \le 0 \text{ for all } a \in A\}.$$

If $K \in \mathcal{K}_s^n$ has interior points, then $K^* \in \mathcal{K}_s^n$ and $(K^*)^* = K$. It follows from (11) and approximation that

$$V_j(K) = V_{n-j-1}(K^*) \tag{12}$$

for $j = 0, \dots, n-1$ and $K \in \mathcal{K}_s^n$. It is consistent with this to define

$$V_{-1}(K) := V_n(K^*).$$

The counterpart to (4), the Gauss-Bonnet formula in spherical space, involves a sequence of intrinsic volumes and not just V_0, namely

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} V_{2i}(K) = \frac{1}{2}\chi(K). \tag{13}$$

For differential-geometric proofs (not restricted to the convex case), we refer to Allendoerfer-Weil [2], Santaló [23], [24]. For $K \in \mathcal{K}_s^n$, (13) gives

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} V_{2i}(K) = \frac{1}{2}, \tag{14}$$

and the Steiner formula (9) for $\epsilon = \pi/2$ (observe that $\sigma_n(K_{\pi/2}) + \sigma_n(K^*) = \sigma_n(\mathbb{S}^n)$) gives

$$\sum_{i=-1}^{n} V_i(K) = 1. \tag{15}$$

From (14) and (15) it follows that

$$\sum_{i=-1}^{n} (-1)^i V_i(K) = 0. \tag{16}$$

Relations (14) and (15) are equivalent to (15) and (16). For spherical polytopes (from which the case of general $K \in \mathcal{K}_s^n$ follows by approximation), (15) has a direct elementary proof (by decomposition of $\mathbb{S}^n$), and an elementary proof of (16) was given by McMullen [19] (using (11); see also Glasauer [10] for more details).

Due to the different form of the Gauss-Bonnet theorem, also the integral-geometric result corresponding to (5) takes a different form. Let $\mathcal{S}_q^n$ denote the space of q-dimensional great subspheres in $\mathbb{S}^n$ (intersections of $\mathbb{S}^n$ with $(q+1)$-dimensional linear subspaces of $\mathbb{E}^{n+1}$), and let ν_q be the rotation invariant probability measure on $\mathcal{S}_q^n$. In view of (5), the functionals defined by

$$U_j(K) := \frac{1}{2} \int_{\mathcal{S}_{n-j}^n} \chi(K \cap S)\, d\nu_{n-j}(S) \tag{17}$$

for $K \in \mathcal{K}_s^n$ and $j = 0, \dots, n$ can also be considered as spherical counterparts of the Euclidean intrinsic volumes (the factor $1/2$ is only for convenience). We have

$$U_j(K) = \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} V_{j+2k}(K). \tag{18}$$

With different notation, this can be found (also for hyperbolic space) in Santaló [25, Section IV.4]. Formula (18) is Corollary 5.2.5 in the thesis of Glasauer [10] (see also [11]), which contains many general results about integral geometry of convex bodies in spherical space. A special one is a spherical counterpart to (6). Let $S \in \mathcal{S}_q^n$, $q \in \{0, \dots, n\}$. Then $S = \mathbb{S}^n \cap L$ with a $(q+1)$-dimensional linear subspace L of $\mathbb{E}^{n+1}$. For $K \in \mathcal{K}_s^n$, the projection $K|S$ of K on S is defined by

$$K|S := S \cap \operatorname{pos}(K \cup L^\perp),$$

where pos denotes the positive hull in $\mathbb{E}^{n+1}$ and $L^\perp$ is the orthogonal complement of L. Defining

$$W_j(K) := \frac{1}{\beta_j} \int_{\mathcal{S}_j^n} \sigma_j(K|S)\, d\nu_j(S),$$

for $K \in \mathcal{K}_s^n$, we have

$$W_j(K) = \sum_{k=j}^{n} V_k(K), \tag{19}$$

by [10, 5.2.11].

It is clear from the definitions that the functionals U_j and W_j are increasing under set inclusion. This is not generally true for the intrinsic volumes V_j (see the discussion in McMullen-Schneider [21, pp. 183 – 184]).

There are now three different sequences of functionals that can be considered as spherical counterparts of the Euclidean intrinsic volumes. Of these, the V_j seem to be the basic ones, since the U_j and W_j are linear combinations of the V_j with nonnegative coefficients. All of these functionals are rotation invariant, continuous with respect to the Hausdorff metric, and additive. In view of Hadwiger's characterization theorem, one may ask whether every function $\varphi : \mathcal{K}_s^n \to \mathbb{R}$ sharing these properties must be of the form $\varphi = \sum_{i=0}^n c_i V_i$ with constant coefficients $c_0, \dots, c_n$. This question has repeatedly been asked ([13, problem 49], [21, problem (15.5)], [20, p. 976]). We have recalled it here, since it is still open, in spite of the immense progress that the theory of valuations on convex bodies has seen in recent years, due to work of S. Alesker, D. Klain, M. Ludwig, and others.

Our main concern here are extremal problems, motivated by the inequalities (7) in the Euclidean case. In noneuclidean spaces, we cannot expect simple inequalities, but we can hope for simple extremal bodies.

Question. Which of the functionals V_j, U_j, W_j in spherical space have the property that they attain an extremum, on the set of convex bodies in $\mathcal{K}_s^n$ of given volume, precisely at balls?

The same question can be posed for the analogues of V_j, U_j in hyperbolic space. The only known result in this direction seems to be the case of

$V_{n-1} = U_{n-1}$, which is the classical isoperimetric problem for the surface area. The extremal property of the ball in noneuclidean spaces was obtained by E. Schmidt [27]; see also Burago-Zalgaller [7]. Also known under the name of 'isoperimetric inequality' in $\mathbb{S}^n$ is the following fact. Let A be a measurable subset. Let $C \subset \mathbb{S}^n$ be a ball with $\sigma_n(A) = \sigma_n(C)$. Then $\sigma_n(A_\epsilon) \geq \sigma_n(C_\epsilon)$. Proofs, by different types of symmetrizations, are found in [27], the appendix of Figiel-Lindenstrauss-Milman [9], Benyamini [4], Schechtman [26]. If we restrict ourselves to convex bodies, this includes, in view of the Steiner formula (9), an extremal property of the ball with respect to certain linear combinations of intrinsic volumes.

The main purpose of this paper is to show that *the functional U_1 attains its minimum on the set of spherical convex bodies of given volume precisely at the balls.* By (18), U_1 is also a sum of certain intrinsic volumes. Since $U_1(K)$ is, up to a factor, the total measure of the spherical hyperplanes meeting K and thus a counterpart to the Euclidean mean width, the result can be considered as a spherical version of the Urysohn inequality. On the other hand, from (18), (14), (15) we have $U_1 = 1/2 - V_{-1}$, or

$$U_1(K) = \frac{1}{2} - V_n(K^*) \tag{20}$$

(which is easy to see directly). Hence, the result can also be considered as a spherical counterpart of the Blaschke-Santaló inequality on the volumes of polar convex bodies.

Let us say that a functional on $\mathcal{K}_s^n$ is *minimal (maximal) at balls* if on each set of bodies in $\mathcal{K}_s^n$ with a fixed value of the volume, the functional attains its minimum (maximum) at balls. By the isoperimetric inequality, V_{n-1} is minimal at balls. The result of this paper says that V_{-1} is maximal at balls. In the special case $n = 3$, relations (14) and (15) give

$$V_0 + V_2 = \frac{1}{2},$$

$$V_{-1} + V_1 + V_3 = \frac{1}{2}.$$

Hence, for $n = 3$, V_0 is maximal at balls, and V_1 is minimal at balls. But already in dimension 4, the situation for V_0, V_1, V_2 is not clear. On the other hand, it seems reasonable to conjecture that U_j is minimal at balls, for $j = 1, \ldots, n-1$ and all n.

For our result, we will give two proofs, of which either one has its intrinsic interest. The first one has the advantage that it works also in hyperbolic space. The second one establishes an interesting connection to the Euclidean Blaschke-Santaló inequality.

2 A proof by two-point symmetrization

We prove the following theorem. As usual, a ball in $\mathbb{S}^n$ will be called a spherical cap. This is the nonempty intersection of $\mathbb{S}^n$ with a closed halfspace of $\mathbb{E}^{n+1}$ (and may be one-pointed).

Theorem. *Let $K \in \mathcal{K}_s^n$. If $C \subset \mathbb{S}^n$ is a spherical cap with $\sigma_n(K) = \sigma_n(C)$, then*

$$U_1(K) \geq U_1(C). \tag{21}$$

Equality holds if and only if K is a spherical cap.

For the proof we use two-point symmetrization, a method which is also known under the names of 'two-point rearrangement', 'compression', or 'polarization'. We learned about this method from a paper of Feige and Schechtman [8]. Its first appearance seems to be in a paper by Wolontis [29]. Benyamini [4] used it for the proof of a spherical isoperimetric inequality. A recent survey of this method (mostly applied to functions) with new applications and many references is given by Brock and Solynin [6].

A two-point symmetrization on $\mathbb{S}^n$ uses a given oriented hyperplane. Let $H \subset \mathbb{E}^{n+1}$ be a hyperplane through 0 together with an orientation, which determines H^+ and H^-, the two closed halfspaces bounded by H. By ρ we denote the reflection with respect to H. Let $A \subset \mathbb{S}^n$. The two-point symmetrization T with respect to H transforms A into the set

$$TA := (A \cap \rho A) \cup [(A \cup \rho A) \cap H^+].$$

Intuitively, T pushes as much of A as possible into H^+ without causing doubly covered points. Writing $A_0 := A \cap \rho A$, we have the disjoint decomposition

$$TA = A_0 \cup \rho[(A \cap H^-) \setminus A_0] \cup [(A \cap H^+) \setminus A_0],$$

which shows that $\sigma_n(TA) = \sigma_n(A)$.

Let $U(\mathcal{K}_s^n)$ denote the system of finite unions of convex bodies in $\mathbb{S}^n$. We extend the definition of the functional U_1 by putting

$$U_1(K) := \frac{1}{2} \int_{\mathcal{S}_{n-1}^n} \chi(K \cap S)\, d\nu_{n-1}(S)$$

for $K \in U(\mathcal{K}_s^n)$. The Euler characteristic χ is needed here as a function defined on $U(\mathcal{K}_s^n)$, which satisfies $\chi(K) = 1$ for $K \in \mathcal{K}_s^n$, $\chi(\emptyset) = 0$, and the additivity property $\chi(A_1 \cup A_1) + \chi(A_1 \cap A_2) = \chi(A_1) + \chi(A_2)$ for $A_1, A_2 \in U(\mathcal{K}_s^n)$. An elementary existence proof, without a recourse to algebraic topology, can be found, e.g., in Groemer [12].

Now let $K \in \mathcal{K}_s^n$ and an oriented hyperplane H through 0 be given. The reflection ρ and the two-point symmetrization T refer to H, H^-, H^+, as above. With $K_0 := K \cap \rho K$, we can write TK in the form

$$TK = K_0 \cup (\rho K \cap H^+) \cup (K \cap H^+).$$

This is in general not a disjoint decomposition, but has the advantage that each of the sets K_0, $\rho K \cap H^+$, $K \cap H^+$ is either empty or a convex body in $\mathbb{S}^n$.

Let L be another hyperplane through 0. The Euler characteristic satisfies the inclusion-exclusion principle, of which we need the special case

$$\begin{aligned}\chi(K_1 \cup K_2 \cup K_2) &= \chi(K_1) + \chi(K_2) + \chi(K_3) - \chi(K_1 \cap K_2) \\ &\quad -\chi(K_1 \cap K_3) - \chi(K_2 \cap K_3) + \chi(K_1 \cap K_2 \cap K_3).\end{aligned}$$

This gives

$$\begin{aligned}&\chi(L \cap TK) \\ &= \chi((L \cap K_0) \cup (L \cap \rho K \cap H^+) \cup (L \cap K \cap H^+)) \\ &= \chi(L \cap K_0) + \chi(L \cap \rho K \cap H^+) + \chi(L \cap K \cap H^+) - \chi(L \cap K_0 \cap H^+) \\ &\quad -\chi(L \cap K_0 \cap H^+) - \chi(L \cap K_0 \cap H^+) + \chi(L \cap K_0 \cap H^+) \\ &= \chi(L \cap K_0) + \chi(L \cap \rho K \cap H^+) + \chi(L \cap K \cap H^+) \\ &\quad -2\chi(L \cap K_0 \cap H^+).\end{aligned}$$

We replace L by ρL and add the two resulting equations, to obtain

$$\begin{aligned}&\chi(L \cap TK) + \chi(\rho L \cap TK) \\ &= \chi(L \cap K_0) + \chi(L \cap \rho K \cap H^+) + \chi(L \cap K \cap H^+) \\ &\quad -2\chi(L \cap K_0 \cap H^+) + \chi(\rho L \cap K_0) + \chi(\rho L \cap \rho K \cap H^+) \\ &\quad +\chi(\rho L \cap K \cap H^+) - 2\chi(\rho L \cap K_0 \cap H^+).\end{aligned}$$

Using further that $\chi(L \cap \rho K \cap H^+) = \chi(\rho L \cap K \cap H^-)$ and $\chi(\rho L \cap K_0) = \chi(L \cap K_0)$, we get

$$\begin{aligned}&\chi(L \cap TK) + \chi(\rho L \cap TK) \\ &= \chi(L \cap K_0) + \chi(\rho L \cap K \cap H^-) + \chi(L \cap K \cap H^+) \\ &\quad -2\chi(L \cap K_0 \cap H^+) + \chi(L \cap K_0) + \chi(L \cap K \cap H^-) \\ &\quad +\chi(\rho L \cap K \cap H^+) - 2\chi(L \cap K_0 \cap H^-) \\ &= 2\chi(L \cap K_0) + \chi(\rho L \cap K) + \chi(\rho L \cap K \cap H) + \chi(L \cap K) \\ &\quad +\chi(L \cap K \cap H) - 2\chi(L \cap K_0) - 2\chi(L \cap K_0 \cap H) \\ &= \chi(L \cap K) + \chi(\rho L \cap K).\end{aligned}$$

Here we have applied the additivity of χ in the form

$$\chi(A \cap H^-) + \chi(A \cap H^+) = \chi(A) + \chi(A \cap H)$$

to $A \in \{L \cap K_0, \rho L \cap K, L \cap K\}$, and we have used that $K \cap H = K_0 \cap H = \rho K \cap H$. The result is

$$\chi(L \cap TK) + \chi(\rho L \cap TK) = \chi(L \cap K) + \chi(\rho L \cap K). \tag{22}$$

The spherically convex hull of $A \subset \mathbb{S}^n$, denoted by $\operatorname{conv} A$, is defined as the intersection of $\mathbb{S}^n$ with $\operatorname{pos} A$, the positive hull of A in $\mathbb{E}^{n+1}$. Next we show that

$$\chi(L \cap \operatorname{conv} TK) \leq \chi(L \cap TK). \tag{23}$$

For the proof, we write

$$\begin{aligned} K_1 &:= K_0 \cup (\rho K \cap H^+), \\ K_2 &:= K_0 \cup (K \cap H^+). \end{aligned}$$

Then K_1, K_2 are either empty or elements of $\mathcal{K}_s^n$. Moreover,

$$TK = K_1 \cup K_2, \qquad K_0 = K_1 \cap K_2$$

and hence

$$\chi(L \cap TK) = \chi(L \cap K_1) + \chi(L \cap K_2) - \chi(L \cap K_0).$$

The set K lies in an open hemisphere, hence in a cap C of (spherical) radius less than $\pi/2$. It follows that $TK \subset TC$, and $TC =: C'$ is a cap congruent to C. Therefore $\operatorname{conv} TK \subset C'$, hence $\operatorname{conv} TK \in \mathcal{K}_s^n$.

Case 1: $\chi(L \cap K_0) = 1$. Then $\chi(L \cap K_1) = 1$, $\chi(L \cap K_2) = 1$, hence $\chi(L \cap TK) = 1$. Since $L \cap \operatorname{conv} TK \neq \emptyset$, we have $\chi(L \cap \operatorname{conv} TK) = 1$.

Case 2: $\chi(L \cap K_0) = 0$.

Subcase a: $\chi(L \cap K_1) = 1$, $\chi(L \cap K_2) = 0$. Then $\chi(L \cap TK) = 1$ and $\chi(L \cap \operatorname{conv} TK) = 1$.

Subcase b: $\chi(L \cap K_2) = 0$, $\chi(L \cap K_1) = 1$. Then $\chi(L \cap TK) = 1$ and $\chi(L \cap \operatorname{conv} TK) = 1$.

Subcase c: $\chi(L \cap K_1) = 0$, $\chi(L \cap K_2) = 0$. Then $\chi(L \cap TK) = 0$. Since L is a hyperplane and $K_1 \cup K_2$ is connected, we have $L \cap \operatorname{conv} TK = \emptyset$ and thus $\chi(L \cap \operatorname{conv} TK) = 0$.

Subcase d: $\chi(L \cap K_1) = 1$, $\chi(L \cap K_2) = 1$. Then $\chi(L \cap TK) = 2$ and $\chi(L \cap \operatorname{conv} TK) = 1$.

This proves the proposition (23) and shows, in particular, that

$$\chi(L \cap \operatorname{conv} TK) < \chi(L \cap TK)$$

holds if and only if

$$L \cap K_1 \neq \emptyset, \quad L \cap K_2 \neq \emptyset, \quad L \cap K_1 \cap K_2 = \emptyset.$$

We write $\operatorname{conv} TK =: K_H$. As a result of (23) (also applied to ρL) and (22), we have

$$\begin{aligned}&\chi(L \cap K_H) + \chi(\rho L \cap K_H)\\ &\leq \chi(L \cap TK) + \chi(\rho L \cap TK) = \chi(L \cap K) + \chi(\rho L \cap K).\end{aligned}$$

Applying the obtained inequality to every great subsphere $S = L \cap \mathbb{S}^n$, we get

$$\begin{aligned}&\int_{\mathcal{S}^n_{n-1}} \chi(S \cap K_H)\, d\nu_{n-1}(S) + \int_{\mathcal{S}^n_{n-1}} \chi(\rho S \cap K_H)\, d\nu_{n-1}(S)\\ &\leq \int_{\mathcal{S}^n_{n-1}} \chi(S \cap K)\, d\nu_{n-1}(S) + \int_{\mathcal{S}^n_{n-1}} \chi(\rho S \cap K)\, d\nu_{n-1}(S).\end{aligned}$$

Here

$$\int_{\mathcal{S}^n_{n-1}} \chi(\rho S \cap K)\, d\nu_{n-1}(S) = \int_{\mathcal{S}^n_{n-1}} \chi(S \cap K)\, d\nu_{n-1}(S) = 2U_1(K),$$

since the measure ν_{n-1} is invariant under the reflection ρ. We conclude that $U_1(K_H) \leq U_1(K)$.

We are now in a position to prove the theorem. The case of volume 0 being trivial, we let $0 < v < \sigma_n(\mathbb{S}^n)/2$ and consider the set of all bodies $K \in \mathcal{K}^n_s$ with $\sigma_n(K) = v$ and minimal $U_1(K)$. By standard arguments, this set is not empty; its elements will be *called extremal* bodies. Let K be an extremal body. Let T be a two-point symmetrization with respect to some oriented hyperplane H. Suppose that TK is not spherically convex. Then the spherically convex hull K_H of TK has larger volume than TK. There exists a convex body $A \subset \operatorname{int} K_H$ with $\sigma_n(A) = v$. From $A \subset \operatorname{int} K_H$ it follows that $U_1(A) < U_1(K_H) \leq U_1(K)$. This contradicts the fact that K is extremal. Hence, TK is spherically convex, and $\sigma_n(TK) = \sigma_n(K)$, $U_1(TK) = U_1(K)$. In particular, we have shown that the extremal body K has the property that every two-point symmetrization preserves the convexity of the body. This implies that K is a spherical cap. Of this fact, Bianchi [5] indicated a proof to the last author. An independent proof and a generalization are due to Aubrun and Fradelizi [3].

Without using the last result, we could also finish the proof in the following alternative way. It has been shown above that the set $\mathcal{E}$ of extremal bodies is closed under two-point symmetrization. First we show that $\mathcal{E}$ contains a spherical cap. For this, we adapt an argument from [4] (see also [26]). Let C be a spherical cap with $\sigma_n(C) = v$. By a standard argument, there exists an element $B \in \mathcal{E}$ for which $\sigma_n(B \cap C)$ is maximal. Suppose that $B \neq C$. Since $\sigma_n(B) = \sigma_n(C)$, there are congruent caps $B_1 \subset B \setminus C$ and $B_2 \subset C \setminus B$. Let H be the oriented hyperplane through 0, with corresponding reflection ρ, so that $\rho B_1 = B_2$ and $B_1 \subset H^-$. The two-point symmetrization T with respect to H gives $\sigma_n(TB \cap C) > \sigma_n(B \cap C)$. Since $TB \in \mathcal{E}$, this is a contradiction. This shows that $B = C$ and thus $C \in \mathcal{E}$. This proves the inequality (21). To show that only spherical caps are extremal, we can argue as follows. Let $K \in \mathcal{E}$, and let $\mathcal{C}$ be the set of all spherically convex bodies that can be obtained from K by the iteration of finitely many two-point symmetrizations. By the same argument as above, one can show that the closure of $\mathcal{C}$ contains a spherical cap. Two-point symmetrization does not change the volume or the surface area. On convex bodies, volume and surface area are continuous. Hence, K has the same volume and surface area as some spherical cap. Among convex bodies in $\mathbb{S}^n$, the caps are the only solutions of the isoperimetric problem. Hence K must be a cap. $\square$

Corollary. *Let $A \subset \mathbb{S}^n$ be a nonempty measurable set. If $C \subset S^n$ is a spherical cap with $\sigma_n(A) = \sigma_n(C)$, then*

$$\sigma_n(A^*) \leq \sigma_n(C^*). \tag{24}$$

Equality holds if A is a spherical cap, and if A is closed and $\sigma_n(A) < \frac{1}{2}\sigma_n(S^n)$, it holds only in this case.

For the proof, we need only consider special sets A. First, inequality (24) and the equality assertion are trivial if $\sigma_n(A^*) = 0$. If $\sigma_n(A^*) > 0$, then A lies in some open hemisphere, hence we can restrict ourselves to this case. Let B be the spherically convex hull of the closure of A. Then $B \in \mathcal{K}_s^n$ and $B^* = A^*$. Hence, we see that it is sufficient to prove the inequality (24) under the assumption that $A \in \mathcal{K}_s^n$. Now the assertion follows from (20) and the Theorem. $\square$

The inequality (24) can be seen in a certain analogy to the Blaschke-Santaló inequality in Euclidean space $\mathbb{E}^n$. Let $K \subset \mathbb{E}^n$ be a convex body with 0 as centroid. Let $K^o := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$ be its polar body. The Blaschke-Santaló inequality says that if E is a centred ellipsoid with $\lambda_n(K) = \lambda_n(E)$, then $\lambda_n(K^o) \leq \lambda_n(E^o)$, with equality if and only if K is a centred ellipsoid. (In Euclidean space, $\lambda_n(E)\lambda_n(E^o)$ is independent of E, but in $\mathbb{S}^n$ there is no similarly elegant relation between $\sigma_n(C)$ and $\sigma_n(C^*)$.) The first proofs of this inequality made use of connections to affine differential geometry and, in particular, the affine isoperimetric inequality, which was obtained by Steiner symmetrization. Later, simpler proofs were found, which make more direct use of Steiner symmetrization and apply, in addition, the Brunn-Minkowski inequality (see the references in [28, p.

425, Note 12]). Neither of these approaches seems to carry over to spherical space.

In the next section, however, we will show that it is possible to deduce the spherical counterpart of the Blaschke-Santaló inequality from the Euclidean one. In contrast to the proof of the theorem given above, this proof does not carry over to the hyperbolic space, so that the method of two-point symmetrization is somewhat superior.

3 Deduction from the Euclidean Blaschke-Santaló inequality

Let $K \in \mathcal{K}_s^n$, and let $C \in \mathcal{K}_s^n$ be a spherical cap with $\sigma_n(K) = \sigma_n(C)$. We want to show that

$$\sigma_n(K^*) \leq \sigma_n(C^*),$$

and that equality holds only if K is a spherical cap. If $\sigma_n(K) = 0$, then the assertion is easy to check. The same is true for the case $n = 1$. Hence, we assume in the following that $\sigma_n(K) > 0$, thus $\operatorname{int} K \neq \emptyset$ (here and in the following, the interior, int, refers to the obvious surrounding space), and $n \geq 2$.

For $e \in -\operatorname{int} K^*$ put $H_e^+ := \{x \in \mathbb{E}^{n+1} : \langle x, e\rangle > 0\}$ and $T_e := e + e^\perp$, so that T_e is the tangent hyperplane of $\mathbb{S}^n$ at e. Then $K \subset \operatorname{int} H_e^+$. We define a map $R_e : \mathbb{S}^n \cap H_e^+ \to T_e$ by $R_e(u) := \langle e, u\rangle^{-1} u$. The Jacobian of this map is given by

$$\mathrm{J}R_e(u) = \langle e, u\rangle^{-(n+1)},$$

hence

$$F(e) := \int_K \langle e, u\rangle^{-(n+1)} \, d\sigma_n(u)$$

is the volume of the set $R_e(K)$ in T_e. It is easy to see that the function $F : -\operatorname{int} K^* \to (0, \infty)$ defined in this way is continuous and that $F(e) \to \infty$ as e approaches the boundary of $-K^*$. Hence, the function F attains a minimum at some point $e \in -\operatorname{int} K^*$. Since all directional derivatives of F at e must vanish, we get

$$\int_K \langle e, u\rangle^{-(n+2)} \langle v, u\rangle \, d\sigma_n(u) = 0 \qquad \text{for } v \in \mathbb{S}^n \cap e^\perp.$$

This implies that the vector

$$\int_K \langle e, u\rangle^{-(n+2)} u \, d\sigma_n(u)$$

is proportional to e and hence equal to $F(e)e$. Consequently, the centroid of $R_e(K)$ in T_e is given by

$$\lambda_n(R_e(K))^{-1} \int_{R_e(K)} x \, d\lambda_n(x) = F(e)^{-1} \int_K \langle e, u\rangle^{-(n+2)} u \, d\sigma_n(u) = e.$$

Now the Blaschke-Santaló inequality, applied in $e^{\perp}$, tells us that

$$\lambda_n(R_e(K))\lambda_n((R_e(K)-e)^o) \leq \kappa_n^2, \tag{25}$$

where the polarity ^{o} is taken in $e^{\perp}$. Equality holds if and only if $R_e(K)$ is an ellipsoid.

We set $S_e := \mathbb{S}^n \cap e^{\perp}$. Let $\alpha : S_e \to (0, \pi/2)$ be the positive continuous function defined by

$$\partial(R_e(K)) = \{e + (\tan\alpha(u))u : u \in S_e\},$$

i.e., $\tan\alpha(\cdot)$ is the radial function of $R_e(K)$ with respect to e in T_e. By σ_{n-1} we denote the spherical Lebesgue measure on S_e. Then

$$\lambda_n(R_e(K)) = \frac{1}{n}\int_{S_e} \tan^n \alpha(u)\, d\sigma_{n-1}(u).$$

We define a map $\tau : (0, \pi/2) \times S_e \to \mathbb{S}^n$ by $\tau(t, u) := (\cos t)e + (\sin t)u$. It has Jacobian $\mathrm{J}\tau(t, u) = \sin^{n-1} t$, hence

$$\sigma_n(K) = \int_{S_e}\int_0^{\alpha(u)} \sin^{n-1} t\, dt\, d\sigma_{n-1}(u).$$

Next we define

$$D(x) := \int_0^x \sin^{n-1} t\, dt, \qquad x \in (0, \pi/2),$$

and set

$$g(s) := D(\arctan s^{1/n}), \qquad s \in (0, \infty).$$

Let h denote the inverse function of g. Then h and D^{-1}, the inverse of D, are related by

$$h(y) = \tan^n D^{-1}(y), \qquad y \in \mathrm{im}(D).$$

Since

$$h'(y) = \frac{n}{\cos^{n+1} D^{-1}(y)},$$

the function h' is strictly increasing, hence h is strictly convex. Therefore, Jensen's inequality gives

$$\begin{aligned}
\frac{\lambda_n(R_e(K))}{\kappa_n} &= \frac{1}{n\kappa_n}\int_{S_e} \tan^n \alpha(u)\, d\sigma_{n-1}(u) \\
&= \frac{1}{n\kappa_n}\int_{S_e} h(D(\alpha(u)))\, d\sigma_{n-1}(u) \\
&\geq h\left(\frac{1}{n\kappa_n}\int_{S_e} D(\alpha(u))\, d\sigma_{n-1}(u)\right) \\
&= h\left(\frac{\sigma_n(K)}{n\kappa_n}\right),
\end{aligned}$$

with equality if and only if α is constant. The assertion concerning the equality case follows from the usual proof of Jensen's inequality in the present particular situation. We conclude that

$$h\left(\frac{\sigma_n(K)}{n\kappa_n}\right) \leq \frac{\lambda_n(R_e(K))}{\kappa_n}, \tag{26}$$

with equality if and only if K is a spherical cap centred at e. Since $-e \in \operatorname{int} K^*$ and $K^* \subset \operatorname{int} H^+_{-e}$, we similarly have

$$h\left(\frac{\sigma_n(K^*)}{n\kappa_n}\right) \leq \frac{\lambda_n(R_{-e}(K^*))}{\kappa_n}, \tag{27}$$

with equality if and only if K^* is a spherical cap centred at $-e$, hence if and only if K is a spherical cap centred at e. Now

$$R_{-e}(K^*) + e = (R_e(K) - e)^o. \tag{28}$$

In fact, both sets lie in $e^\perp$, and for $y \in e^\perp$ we have

$$\begin{aligned}
& y \in (R_e(K) - e)^o \\
& \Leftrightarrow \langle y, x - e\rangle \leq 1 \; \forall\, x \in R_e(K) \Leftrightarrow \langle y, x\rangle \leq 1 \; \forall\, x \in R_e(K) \\
& \Leftrightarrow \langle y, \langle e, k\rangle^{-1} k\rangle \leq 1 \; \forall\, k \in K \Leftrightarrow \langle y - e, k\rangle \leq 0 \; \forall\, k \in K \\
& \Leftrightarrow y \in R_e(K^*) + e.
\end{aligned}$$

From (27) and (28) we get

$$h\left(\frac{\sigma_n(K^*)}{n\kappa_n}\right) \leq \frac{\lambda_n((R_e(K) - e)^o)}{\kappa_n}, \tag{29}$$

with equality if and only if K is a spherical cap centred at e.

Let $C \subset \mathbb{S}^n$ be a spherical cap centred at e with $\sigma_n(C) = \sigma_n(K)$. Then (26), together with the assertion on the equality case, gives

$$\frac{\lambda_n(R_e(C))}{\kappa_n} = h\left(\frac{\sigma_n(C)}{n\kappa_n}\right) = h\left(\frac{\sigma_n(K)}{n\kappa_n}\right) \leq \frac{\lambda_n(R_e(K))}{\kappa_n} \tag{30}$$

Using (29), (25), (30), the equality case of the Blaschke-Santaló inequality, and the equality case of (29), we deduce that

$$\begin{aligned}
h\left(\frac{\sigma_n(K^*)}{n\kappa_n}\right) &\leq \frac{\lambda_n\left((R_e(K) - e)^o\right)}{\kappa_n} \\
&\leq \frac{\kappa_n}{\lambda_n(R_e(K))} \\
&\leq \frac{\kappa_n}{\lambda_n(R_e(C))} \\
&= \frac{\lambda_n\left((R_e(C) - e)^o\right)}{\kappa_n} \\
&= h\left(\frac{\sigma_n(C^*)}{n\kappa_n}\right).
\end{aligned}$$

This implies the required inequality, since h is strictly increasing. If equality holds in the inequality $\sigma_n(K^*) \leq \sigma_n(C^*)$ thus obtained, then equality must hold in (29), which means that K is a spherical cap. $\square$

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