Majorizing Measure Bounds for Processes with Mixed Exponential Tails

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Abstract

If $\{X_t\}_{t\in T}$ is a process with tail probability

$$\Pr(|X_t - X_s| > u) \leq \exp\left(-\min\left\{\frac{A_i u^{\alpha_i}}{\|t-s\|_i^{\alpha_i}} : 1 \leq i \leq d\right\}\right),$$

where A_i and α_i are positive constants, and $\|\cdot\|_i$, $1 \leq i \leq d$, represent d different norms. Suppose $\alpha_1 = \min\{\alpha_i\}$. Then there exists a constant K depending only on d and α_i's, such that

$$\left\|\sup_{t\in T}|X_t|\right\|_{\psi_{\alpha_1}} \leq \|X_{t_0}\|_{\psi_{\alpha_1}} + K\sum_{i=1}^{d} A_i^{-1/\alpha_i}\gamma_{\alpha_i}(T, \|\cdot\|_i),$$

where $t_0 \in T$, and γ_{α_i} are the majorizing measures of T.

This is an extension of the corresponding result in [T2], where $d = 2$ was considered. Some techniques we use here are similar to that of [LT], though the latter contains an error. After this paper had been written, the author learned from Talagrand that the result had appeared in [M]. The author have not got the access to [M]. But according to Talagrand, the method of [M] is a rewriting of [T2]. So, our methods should be different. Also, we deal with Orliz norm, which is a stronger result.

A few weeks after a preprint of this paper was sent to Talagrand, the author received a preprint [T4] from him, which contains a different (and simple !) proof of this result. The author noticed and informed Talagrand that by regrouping the summation in our approach, the main result of [T4] can be recovered. However, the approach in [T4] is superior. So the author decide not to submit this paper for publication.

Lemma 1 seems to be new. It gives a quick proof of the theorem for the case $d = 1$ with Orliz norm. (For example, the sub-Gaussian case.) This is the main reason to make this paper accessible to public.

0.1 Introduction

Consider a process $\{X_t\}_{t\in T}$ with tail probability

$$\Pr(|X_t - X_s| > u) \le \exp\left(-\min\left\{\frac{A_i u^{\alpha_i}}{\|t-s\|_i^{\alpha_i}} : 1 \le i \le d\right\}\right), \qquad (1)$$

where A_i and α_i are positive constants, and $\|\cdot\|_i$, $1 \le i \le d$, represent d different norms. It is of interest to study $\sup_{t\in T} |X_t|$. In this note, we will prove

Theorem 1 *If $\{X_t\}_{t\in T}$ is a process satisfying (1) with $\alpha_1 = \min\{\alpha_i : 1 \le i \le d\}$, then there exists a constant K depending only on d and α_i's, such that*

$$\left\|\sup_{t\in T} |X_t|\right\|_{\psi_{\alpha_1}} \le \quad \|X_{t_0}\|_{\psi_{\alpha_1}} + K \sum_{i=1}^{d} A_i^{-1/\alpha_i} \gamma_{\alpha_i}(T, \|\cdot\|_i),$$

where $t_0 \in T$, and γ_{α_i} are the majorizing measures of T. (See the definition below.)

For a bounded (not a singleton) set T in a normed space $(X, \|\cdot\|)$, the majorizing measure $\gamma_\alpha(T, \|\cdot\|)$ is defined by

$$\gamma_\alpha(T, \|\cdot\|) := \inf_\mu \sup_{t\in T} \int_0^\infty \left[\log \frac{1}{\mu(B(t,\varepsilon))}\right]^{1/\alpha} d\varepsilon, \tag{2}$$

where $B(t, \varepsilon)$ is the ball in X with center $t \in T$ and radius $\varepsilon > 0$; μ is a probability measure on T; and $\mu(B(t,\varepsilon))$ is understood as $\mu(B(t,\varepsilon) \cap T)$. The infimum in (2) is taken over all probability measure μ. We note that the right side of (2) remains the same if we replace T by any dense subset $T' \subset T$. It can easily be proved that

$$\frac{1}{r}\gamma_\alpha(T) \le \sup_{t\in T} \sum_{k=b}^{\infty} r^{-k} \left[\log \frac{1}{\mu(B(t, r^{-k}))}\right]^{1/\alpha} \le \frac{r}{r-1}\gamma_\alpha(T)$$

for $r > 1$, where b is an integer satisfying $r^{-b} \ge \operatorname{diam}(T)$.

0.2 An Equivalent Definition

In this section, we introduce "net ordering" to explain majorizing measures. The idea of net-ordering was developed by analyzing [T1] when the author was assigned as a course project for a probability course in early 96. Later on, we learned that [T3] more or less contained a similar idea, but not the same. We believe this "net-ordering" enables one to understand majorizing measures better, at least in the author's case. We also had realized that by regrouping the summands in the definition, one could obtain a simpler form as in [T4]. However, we will then lose some useful information about the set. Also, the current formulation itself well explains the relation between and Dudley's metric entropy condition.

For $\varepsilon > 0$, an ε-net $N(\varepsilon)$ of T is a set such that for every $t \in T$, there exists a point $s = s(t) \in N(\varepsilon)$ satisfying $\|t - s\| \le \varepsilon$.

An ordering of a finite set S is a one-to-one map from S onto $\{1, 2, \ldots, |S|\}$, where $|S|$ is the cardinality of S.

For $r > 1$ and integer k, let N_k be an r^{-k}-net of T, and let I_k be an ordering of N_k. Because T is bounded, there exists the largest integer b, such that N_b contains only one element, say t_0.

Fix $n \geq b$. For each $t \in N_n$. We denote $\pi_n(t,n) = t$, and select $\pi_{n-1}(t,n)$, $\pi_{n-2}(t,n)$, …, $\pi_k(t,n)$, … in order, so that $\pi_k(t,n) \in N_k$,

$$\|\pi_k(t,n) - t\| \leq r^{-k},$$

and $I_k(\pi_k(t,n))$ is as small as possible. The procedure stops when we reach $\pi_b(t,n) = t_0$. Thus we have the expansion

$$t = t_0 + \sum_{k=b}^{n-1} \pi_{k+1}(t,n) - \pi_k(t,n). \tag{3}$$

In general, $\pi_k(t,n)$ may depend on n. When there is no confusion, we simply denote it by $\pi_k(t)$.

We also denote

$$M_\alpha(r,n) = \sup_{t \in N_n} \sum_{k=b}^{n-1} r^{-k} \left[\log I_k(\pi_k(t))\right]^{1/\alpha}.$$

Clearly, M_α depends on nets N_k, and their orderings I_k, $b \leq k \leq n$. Define

$$O_\alpha(T,r) = \inf \sup_{n \geq b} M_\alpha(r,n), \tag{4}$$

where the infimum is taken over all choices of nets N_k and all the orderings of these nets.

The following equivalent condition will be used later.

Theorem 2 *There exist a constant K depending only on r, $r \geq 3$, and α such that*

$$\frac{1}{K}\gamma_\alpha(T) \leq O_\alpha(T,r) \leq K\gamma_\alpha(T).$$

Proof: (i) Without loss of generality, we assume T be compact. Suppose $\gamma_\alpha(T) < \infty$. There exists a probability measure μ on T, such that

$$\sup_{t \in T} \sum_{k \geq b} r^{-k} \left[\log \frac{1}{\mu(B(t, r^{-k-1}))}\right]^{1/\alpha} \leq r^2 \gamma_\alpha(T).$$

For each $k \geq b$, we let N_k consist of elements $s_1, s_2, \ldots, s_i, \ldots$ chosen successively, such that

$$\|s_i - s_j\| \geq r^{-k} \tag{5}$$

for $i \neq j$, and $\mu(B(s_i, r^{-k-1}))$ is as large as possible. (The maximum can be reached by assuming T be compact.) Let I_k be the natural ordering of N_k. That is $I_k(s_i) = i$.

For each $n \geq b$ and $t \in N_n$, by (3) we have the expansion

$$t = t_0 + \sum_{k=b}^{n-1} \pi_{k+1}(t) - \pi_k(t),$$

where $\pi_b(t) = t_0$, ..., $\pi_k(t)$, ..., $\pi_n(t) = t$ satisfy $\pi_k(t) \in N_k$,

$$\|\pi_k(t) - t\| \leq r^{-k}$$

and $I_k(\pi_k(t))$ is as small as possible.

Suppose $I_k(\pi_k(t)) = m$. By the selection of $\pi_k(t)$, if $s \in N_k$ satisfies $I_k(s) < m$, then

$$\|s - t\| > r^{-k}. \tag{6}$$

Therefore, by the selection of $\pi_k(t)$, we have

$$\mu(B(t, r^{-k-1})) \leq \mu(B(\pi_k(t), r^{-k-1})). \tag{7}$$

We also note that by (5), the sets $B(s, r^{-k-1})$, $I_k(s) \leq m$, are pairwise disjoint, and each of them satisfies

$$\mu(B(s, r^{-k-1})) \geq \mu(B(\pi_k(t), r^{-k-1})). \tag{8}$$

Therefore

$$\mu(B(\pi_k(t), r^{-k-1})) \leq \frac{1}{m} = \frac{1}{I_k(\pi_k(t))}.$$

Plug into (7), we obtain

$$I_k(\pi_k(t)) \leq \frac{1}{\mu(B(t, r^{-k-1}))}.$$

Therefore

$$
\begin{aligned}
M_\alpha(r,n) &= \sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[\log I_k(\pi_k(t))\right]^{1/\alpha} \\
&\le \sup_{t\in N_n}\sum_{k\ge b} r^{-k}\left[\log\frac{1}{\mu(B(t,r^{-k-1}))}\right]^{1/\alpha} \\
&\le r^2\gamma_\alpha(T),
\end{aligned}
$$

which implies $O_\alpha(T,r)\le K\gamma_\alpha(T)$ for $K=r^2$.

(ii) Suppose $O_\alpha(T,r)<\infty$. There exist nets N_k and their orderings I_k, such that for all $n>b$,

$$
\sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[\log I_k(\pi_k(t))\right]^{1/\alpha}\le 2O_\alpha(T,r).
$$

We take a discrete probability measure μ such that for each $n\ge b$ and $t\in N_n$,

$$
\mu(\{\pi_k(t)\})\ge\left(\frac{6}{\pi^2}\right)^2\left[\frac{1}{(k-b+1)I_k(\pi_k(t))}\right]^2.
$$

Because $\|t-\pi_k(t)\|\le r^{-k}$, we have $\pi_k(t)\in B(t,r^{-k})$. Therefore

$$
\begin{aligned}
&\sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[\log\frac{1}{\mu(B(t,r^{-k}))}\right]^{1/\alpha} \\
\le\ &\sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[\log\frac{1}{\mu(\{\pi_k(t)\})}\right]^{1/\alpha} \\
\le\ &\sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[2\log\left(\frac{6}{\pi^2}\right)+2\log(k-b+1)+2\log I_k(\pi_k(t))\right]^{1/\alpha} \\
\le\ &K_1 r^{-b}+K_2\sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[\log I_k(\pi_k(t))\right]^{1/\alpha} \\
\le\ &KO_\alpha(T,r),
\end{aligned}
$$

where in the last inequality we used the assumption that T is not a singleton.

Because $\bigcup_{n\ge b} N_n$ is dense in T, we have

$$
\sup_{t\in T}\sum_{k\ge b} r^{-k}\left[\frac{1}{\mu(B(t,r^{-k}))}\right]^{1/\alpha}=\sup_{n\ge b}\sup_{t\in N_n}\sum_{k=b}^{n-1} r^{-k}\left[\log\frac{1}{\mu(B(t,r^{-k}))}\right]^{1/\alpha}.
$$

Thus $\gamma_\alpha(T) \le KO_\alpha(T,r)$. ■

0.3 Nets and Their Orderings

Fix r so that $r \ge 3^{\alpha_i}$ for all $1 \le i \le d$. By homogeneity, we can assume that $A_i = 1$ and $\gamma_{\alpha_i}(T, \|\cdot\|_i) \le r^{-3/\alpha_i}$ for all $1 \le i \le d$. By Theorem 2, for any $1 \le i \le d$ and $k \ge 0$, there exists a r^{-k/α_i}-net $N_k^{(i)}$ of $(T, \|\cdot\|_i)$ and an ordering $I_k^{(i)}$, such that, for all $1 \le i \le d$ and $n > 0$

$$\sup_{t \in N_n^{(i)}} \sum_{k=0}^{n-1} r^{-k/\alpha_i} \left[\log I_k^{(i)}(\pi_k^{(i)}(t))\right]^{1/\alpha_i} \le r^{2/\alpha_i}\gamma_{\alpha_i}(Y, \|\cdot\|_i). \tag{9}$$

Denote

$$\beta_k^{(i)}(t) = \sum_{j=1}^{k+1} \left(\log I_j^{(i)}(\pi_j^{(i)}(t))\right)^{1/\alpha_i}.$$

Then, (9) implies that

$$\sup_{t \in N_n^{(i)}} \sum_{k=0}^{n-1} r^{-k/\alpha_i}\beta_k^{(i)}(t) \le r^{3/\alpha_i}\gamma_{\alpha_i}(T, \|\cdot\|) \le 1 \tag{10}$$

for all $n > 0$ and $1 \le i \le d$.

Fix $n \ge 0$. For each $1 \le i \le d$, consider $N_{2n}^{(i)}$. For each $u_i \in N_{2n}^{(i)}$ and $0 \le k \le n$, let $m_i(k, u_i)$ be an integer depending on k and u_i, and to be fixed later, such that

$$k \le m_i(k, u_i) \le 2k,$$

and

$$m_i(k, u_i) \ge m_i(k-1, u_i) \text{ for } 0 < k \le n.$$

Denote

$$B_k^{(i)}(u_i) = \left\{t \in T : \left\|t - \pi_{m_i(u_i,k)}^{(i)}(u_i)\right\|_i \le 2r^{-m_i(k,u_i)/\alpha_i}\right\}.$$

Because

$$\begin{aligned} B_k^{(i)}(u_i) &\supset \left\{t \in T : \|t - u_i\|_i \le r^{-m_i(k,u_i)/\alpha_i}\right\} \\ &\supset \left\{t \in T : \|t - u_i\|_i \le r^{-2n/\alpha_i}\right\}, \end{aligned}$$

we have $\bigcup_{u_i\in N_{2n}^{(i)}} B_k^{(i)}(u_i)$ covers T.

For each $u_i \in N_{2n}^{(i)}$, $1 \le i \le d$. If the set $\bigcap_{i\le d} B_k^{(i)}(u_i)$ is non-empty, we randomly select a point from this set, and denote it by $\pi_k(u_1, u_2, \cdots, u_d)$. We further assume that if

$$\bigcap_{i\le d} B_k^{(i)}(u_i) = \bigcap_{i\le d} B_k^{(i)}(u_i')$$

then

$$\pi_k(u_1, u_2, \cdots, u_d) = \pi_k(u_1', u_2', \cdots, u_d').$$

Thus, $\pi_k(u_1, u_2, \cdots, u_d)$ is determined by $m_i(k, u_i)$ and $\pi^{(i)}_{m_i(u_i,k)}(u_i)$, $1 \le i \le d$.

Denote

$$N_k = \left\{\pi_k(u_1, u_2, \cdots, u_d) : u_i \in N_{2n}^{(i)}, 1 \le i \le d\right\}.$$

Next, we give an ordering to N_k. We define I_k so that if

$$\pi_k(u_1, u_2, \ldots, u_d) \in N_k \text{ and } \pi_k(u_1', u_2', \ldots, u_d') \in N_k$$

satisfy

$$\prod_{i=1}^d I^{(i)}_{m_i(k,u_i)}(\pi^{(i)}_{m_i(k,u_i)}(u_i)) > \prod_{i=1}^d I^{(i)}_{m_i(k,u_i')}(\pi^{(i)}_{m_i(k,u_i')}(u_i')),$$

then

$$I_k(\pi_k(u_1, u_2, \cdots, u_d)) > I_k(\pi_k(u_1', u_2', \ldots, u_d')).$$

We prove that under such an ordering, we have

$$I_k(\pi_k(u_1, u_2, \cdots, u_d)) \le \left[(k+1)\prod_{i=1}^d I^{(i)}_{m_i(k,u_i)}(\pi^{(i)}_{m_i(k,u_i)}(u_i))\right]^d. \tag{11}$$

To see this, let

$$U = \prod_{i=1}^d I^{(i)}_{m_i(k,u_i)}(\pi^{(i)}_{m_i(k,u_i)}(u_i)),$$

and suppose

$$\prod_{i=1}^d I^{(i)}_{m_i(k,u_i')}(\pi^{(i)}_{m_i(k,u_i')}(u_i')) \le U. \tag{12}$$

For each $i \le d$, $m_i(k, u'_i)$ has no more than $k+1$ choices; and when $m_i(k, u'_i)$ fixed, $\pi_{m_i(k,u'_i)}(u'_i)$ has no more than U choices. Because $\pi_k(u'_1, u'_2, \cdots, u'_d)$ is determined by $m_i(k, u'_i)$ and $\pi^{(i)}_{m_i(u'_i,k)}(u'_i)$, $1 \le i \le d$, we conclude that there are no more than $(k+1)^d U^d$ different $\pi_k(u'_1, u'_2, \cdots, u'_d)$ satisfying (12). This implies (11).

Fix $n \ge 0$. For each $t = \pi_n(u_1, u_2, \cdots, u_d) \in N_n$, denote

$$\pi_k(t) = \pi_k(u_1, u_2, \cdots, u_d) \text{ for } 0 \le k \le n.$$

Because we can assume $N_0^{(i)}$ be a singleton for all $1 \le i \le d$, we can assume that $\pi_0(t)$ does not depend on t. We denote $\pi_0(t) = t_0$. Now we can write

$$\begin{aligned} & \left\| \sup_{t \in N_n} |X_t| \right\|_{\psi_{\alpha_1}} \\ \le\ & \|X_{t_0}\|_{\psi_{\alpha_1}} + \left\| \sup_{t \in N_n} \sum_{k=0}^{n-1} \left| X_{\pi_{k+1}(t)} - X_{\pi_k(t)} \right| \right\|_{\psi_{\alpha_1}} \\ \le\ & \|X_{t_0}\|_{\psi_{\alpha_1}} + \sup_{t \in N_n} \sum_{k=0}^{n-1} \varepsilon_k(t) + \left\| \sup_{t \in N_n} \sum_{k=0}^{n-1} |Y_k(t)| \right\|_{\psi_{\alpha_1}} \end{aligned} \tag{13}$$

where

$$Y_k(t) = \left(\left| X_{\pi_{k+1}(t)} - X_{\pi_k(t)} \right| - \varepsilon_k(t) \right)^+ .$$

To estimate the last term in (13), we use the following lemma.

0.4 A Lemma

Lemma 1 *Let ψ be a Young function satisfying*

$$\psi^{-1}(xy) \le K \psi^{-1}(x) \psi^{-1}(y)$$

and

$$\psi^{-1}(x^2) \le K \psi^{-1}(x)$$

for some constant K. Let F_k be a finite set of random variables of ψ-norm bounded by 2^{-k}, and let J_k be an ordering of F_k. Suppose S is a subset of

$\left\{\sum_{k\geq 0} Y_k : Y_k \in F_k, k \geq 0\right\}$. *Then*

$$\left\| \sup_S \sum_{k\geq 0} |Y_k| \right\|_\psi \leq C_\psi \sup_S \sum_{k\geq 0} \psi^{-1}(J_k(Y_k)) \left\| Y_k \right\|_\psi ,$$

where C_ψ *is a constant depending only on* ψ*.*

Remark 1 *The following weaker result was proved in [AG]:*

$$\left\| \sup_{k\geq 1} |Z_k| \right\|_\psi \leq C_\psi \sup_{k\geq 1} \left\| \psi^{-1}(k) \cdot Z_k \right\|_\psi . \tag{14}$$

Proof: Without loss of generality, we assume

$$C_\psi \sup_S \sum_{k\geq 0} \psi^{-1}(J_k(Y_k) + k + 1) \left\| Y_k \right\|_\psi \leq 1,$$

where C_ψ is a constant to be determined later. For $u \geq 1/2$,

$$\begin{aligned} A := & \left\{ \psi\left(\sup_S \sum_{k\geq 0} |Y_k| \right) > u \right\} \\ = & \bigcup_S \left\{ \sum_{k\geq 0} |Y_k| > \psi^{-1}(u) \right\} \\ \subset & \bigcup_S \bigcup_k \left\{ |Y_k| > C_\psi \psi^{-1}(J_k(Y_k) + k + 1) \left\| Y_k \right\|_\psi \psi^{-1}(u) \right\} \\ = & \bigcup_k \bigcup_{F_k} \left\{ |Y_k| > C_\psi \psi^{-1}(J_k(Y_k) + k + 1) \left\| Y_k \right\|_\psi \psi^{-1}(u) \right\} . \end{aligned}$$

Therefore

$$\begin{aligned} \Pr(A) \leq & \sum_{k\geq 0} \sum_{F_k} \Pr\left(|Y_k| > C_\psi \psi^{-1}(J_k(Y_k) + k + 1) \left\| Y_k \right\|_\psi \psi^{-1}(u) \right) \\ \leq & \sum_{k\geq 0} \sum_{F_k} \Pr\left(\psi\left(\frac{|Y_k|}{\left\| Y_k \right\|_\psi} \right) > \psi\left(C_\psi \psi^{-1}(J_k(Y_k) + k + 1) \psi^{-1}(u) \right) \right) \\ \leq & \sum_{k\geq 0} \sum_{F_k} \frac{1}{\psi\left(C_\psi \psi^{-1}(J_k(Y_k) + k + 1) \psi^{-1}(u) \right)} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k\geq 0}\sum_{F_k} \frac{1}{16(J_k(Y_k)+k+1)^4u^2} \quad \text{(providing } C_\psi \text{ large enough)} \\
&\leq \frac{1}{4u^2}.
\end{aligned}
$$

Thus,

$$
E\psi\left(\sup_S \sum_{k\geq 0}|Y_k|\right) \leq \frac{1}{2} + \int_{1/2}^{\infty} \frac{1}{4u^2}du = 1
$$

proving the lemma. ■

0.5 Proof of the Theorem

Now we turn back to (13). For each $t = \pi_n(u_1, \cdots, u_d) \in N_n$. We choose a small $\varepsilon_k(t)$, such that $\|Y_k(t)\|_{\psi_{\alpha_1}} \leq Kr^{-k/\alpha_1}$ for some constant K depending only on α_i's. A proper choice is

$$
\varepsilon_k(t) = \sum_{i=2}^{d} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)}.
$$

(When $\alpha_i = \alpha_1$, we omit the corresponding term in the summation.) In fact, under such a choice

$$
\begin{aligned}
&\Pr\left(|Y_k(t)| \geq s\right) \\
\leq\ & \Pr\left(\left|X_{\pi_{k+1}(t)} - X_{\pi_k(t)}\right| \geq s + \sum_{i=2}^{d} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)}\right) \\
\leq\ & \exp\left(-\min\left\{\frac{s^{\alpha_1}}{\|\pi_{k+1}(t)-\pi_k(t)\|_i^{\alpha_1}}, \frac{r^{k-m_i(k,u_i)}s^{\alpha_1}}{\|\pi_{k+1}(t)-\pi_k(t)\|_i^{\alpha_i}}, 2\leq i\leq d\right\}\right).
\end{aligned}
$$

Because

$$
\begin{aligned}
&\|\pi_k(t) - u_i\|_i \\
\leq\ & \left\|\pi_k(t) - \pi^{(i)}_{m_i(k,u_i)}(u_i)\right\|_i + \left\|\pi^{(i)}_{m_i(k,u_i)}(u_i) - u_i\right\|_i \\
\leq\ & 2r^{-m_i(k,u_i)/\alpha_i} + r^{-m_i(k,u_i)/\alpha_i} = 3r^{-m_i(k,u_i)/\alpha_i},
\end{aligned}
$$

and similarly,

$$
\|\pi_{k+1}(t) - u_i\|_i \leq 3r^{-m_i(k+1,u_i)/\alpha_i} \leq 3r^{-m_i(k,u_i)/\alpha_i},
$$

we have

$$\|\pi_{k+1}(t)-\pi_k(t)\|_i \le 6r^{-m_i(k,u_i)/\alpha_i}.$$

Therefore

$$\Pr\left(|Y_k(t)|\ge s\right)\le \exp\left(-\min_{1\le i\le d}\left\{6^{-\alpha_i}r^k s^{\alpha_1}\right\}\right),$$

which implies that

$$\|Y_k(t)\|_{\psi_{\alpha_1}}\le Kr^{-k/\alpha_1}. \tag{15}$$

Next, we define the ordering of $Y_k(t)$ so that if $Y_k(t)$ and $Y_k(t')$ satisfy

$$I_{k+1}(\pi_{k+1}(t))\cdot I_k(\pi_k(t)) > I_{k+1}(\pi_{k+1}(t'))\cdot I_k(\pi_k(t')),$$

then $J_k(Y_k(t)) > J_k(Y_k(t'))$. Under such definition, we have

$$J_k(Y_k(t)) \le \left[I_{k+1}(\pi_{k+1}(t))\cdot I_k(\pi_k(t))\right]^2. \tag{16}$$

Applying Lemma 1, and then using (15), (16) and (11), we can bound the last two terms of (13) by

$$\begin{aligned}
&\sum_{i=2}^{d}\sup_{t\in N_n}\sum_{k=0}^{n-1} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)}\\
&+K_1\sup_{t\in N_n}\sum_{k=0}^{n-1} r^{-k/\alpha_1}\left(\log J_k(Y_k(t))\right)^{1/\alpha_1}\\
\le\ &\sum_{i=2}^{d}\sup_{t\in N_n}\sum_{k=0}^{n-1} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)} + K_2 r\sum_{k=0}^{n-1} r^{-k/\alpha_1}[\log(k+1)]^{1/\alpha_1}\\
&+K_2\sum_{i=2}^{d}\sup_{t\in N_n}\sum_{k=0}^{n-1} r^{-k/\alpha_1}\left[\log I^{(i)}_{m_i(k+1,u_i)}(\pi^{(i)}_{m_i(k+1,u_i)}(u_i))\right]^{1/\alpha_1}\\
&+K_2\sup_{t\in N_n}\sum_{k=0}^{n-1} r^{-k/\alpha_1}\left[\log I^{(1)}_{m_1(k+1,u_1)}(\pi^{(1)}_{m_1(k+1,u_1)}(u_1))\right]^{1/\alpha_1}\\
\le\ &\sum_{i=2}^{d}\sup_{u_i\in N^{(i)}_{2n}}\sum_{k=0}^{n-1} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)} + K_3\\
&+K_3\sum_{i=2}^{d}\sup_{u_i\in N^{(i)}_{2n}}\sum_{k=0}^{n-1} r^{-k/\alpha_1}\left[\log I^{(i)}_{m_i(k+1,u_i)}(\pi^{(i)}_{m_i(k+1,u_i)}(u_i))\right]^{1/\alpha_1}
\end{aligned}$$

$$
\begin{aligned}
&+K_3 \sup_{u_1 \in N_{2n}^{(1)}} \sum_{k=0}^{n-1} r^{-k/\alpha_1} \left[\log I^{(1)}_{m_1(k+1,u_1)}(\pi^{(1)}_{m_1(k+1,u_1)}(u_1))\right]^{1/\alpha_1} \\
\leq\ & \sum_{i=2}^{d} \sup_{u_i \in N_{2n}^{(i)}} \sum_{k=0}^{n-1} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)} + K_3 \qquad (17) \\
&+K_3 \sum_{i=2}^{d} \sup_{u_i \in N_{2n}^{(i)}} \sum_{k=0}^{n-1} r^{-k/\alpha_1} \left(\beta^{(i)}_{m_i(k,u_i)}(u_i)\right)^{\alpha_i/\alpha_1} \qquad (18) \\
&+K_3 \sup_{u_1 \in N_{2n}^{(1)}} \sum_{k=0}^{n-1} r^{-k/\alpha_1} \left(\log I^{(1)}_{m_1(k+1,u_1)}(\pi^{(1)}_{m_1(k+1,u_1)}(u_1))\right)^{1/\alpha_1}. \qquad (19)
\end{aligned}
$$

Now it is the time to choose $m_i(k,u_i)$. We choose $m_1(k,u_i) = k$. By (9), we can bound (19) by K. For $2 \leq i \leq d$, we choose $m_i(k,u_i)$ so as to "minimize"

$$
r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)} + r^{-k/\alpha_1} \left(\beta^{(i)}_{m_i(k,u_i)}(u_i)\right)^{\alpha_i/\alpha_1},
$$

and keep the conditions $k \leq m_i(k,u_i) \leq 2k$ and $m_i(k,u_i) \geq m_i(k-1,u_i)$ in mind. A proper choice is to let $m_i(k,u_i)$ be the largest integer, less than or equal to $2k$, such that for $j \leq m_i(k,u_i)$,

$$
r^{-k/\alpha_1} \left(\beta^{(i)}_j(u_i)\right)^{\alpha_i/\alpha_1} \leq r^{-(j-k)/(\alpha_i-\alpha_1)},
$$

or equivalently,

$$
r^{-j/\alpha_i} \beta^{(i)}_j(u_i) \leq r^{-(j-k)/(\alpha_i-\alpha_1)}.
$$

Thus, (17) + (18) can be bounded by

$$
\begin{aligned}
& \sum_{i=2}^{d} \sup_{u_i \in N_{2n}^{(i)}} \sum_{k=0}^{n-1} r^{-(m_i(k,u_i)-k)/(\alpha_i-\alpha_1)} \\
\leq\ & K \sum_{i=2}^{d} \sup_{u_i \in N_{2n}^{(i)}} \sum_{l=0}^{2n-2} \sum_{k \in D_l^{(i)}} r^{-(l-k)/(\alpha_i-\alpha_1)}, \qquad (20)
\end{aligned}
$$

where $D_l^{(i)} = \{k : m_i(k,u_i) = l\}$. Let $k_0 = \min\{k : k \in D_l^{(i)}\}$, we have

$$
\sum_{k \in D_l^{(i)}} r^{-(l-k)/(\alpha_i-\alpha_1)} \leq K r^{-(m_i(k_0,u_i)-k_0)/(\alpha_i-\alpha_1)}.
$$

On the other hand,

$$
\begin{aligned}
r^{-(m_i(k_0,u_i)+1-k_0)/(\alpha_i-\alpha_1)} &< r^{-(m_i(k_0,u_i)+1)/\alpha_i}\beta^{(i)}_{m_i(k_0,u_i)+1}(u_i) + r^{-k_0/(\alpha_i-\alpha_1)} \\
&\leq r^{-(l+1)/\alpha_i}\beta^{(i)}_{l+1}(u_i) + r^{-l/2(\alpha_i-\alpha_1)},
\end{aligned}
$$

Thus, we can bound (20) by

$$
\sum_{i=2}^{d} \sup_{u_i \in N^{(i)}_{2n}} r^{1/(\alpha_i-\alpha_1)} \sum_{l=0}^{2n-2} \left[r^{-(l+1)/\alpha_i}\beta^{(i)}_{l+1}(u_i) + r^{-l/2(\alpha_i-\alpha_1)} \right].
$$

The theorem follows by applying (10).
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0.6 References

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