ME 548 Elasticity Mid-Term Exam

Open book, open notes

Due: Monday, Oct. 28, 6 pm PST

To be submitted by email.

1. Scalar and vector field functions are given by $\phi = x^2 - y^2$ and $\boldsymbol{u} = 2xe_1 + 3yze_2 + xye_3$. Calculate the following expressions, $\nabla\phi$, $\nabla^2\phi$, $\nabla \cdot \boldsymbol{u}$, $\nabla\boldsymbol{u}$, $\nabla \times \boldsymbol{u}$.

2. Calculate the quantities $\nabla \cdot \boldsymbol{u}$, $\nabla \times \boldsymbol{u}$, $\nabla^2\boldsymbol{u}$, $\nabla\boldsymbol{u}$, $tr(\nabla\boldsymbol{u})$ for the following Cartesian vector fields:

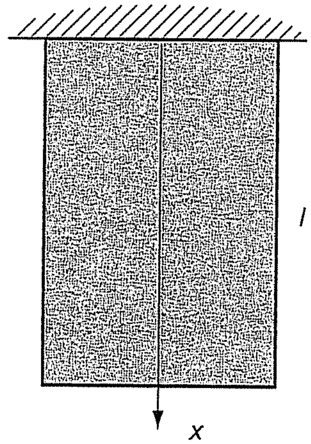
   (a) $\boldsymbol{u} = x_1\boldsymbol{e}_1 + x_1x_2\boldsymbol{e}_2 + 2x_1x_2x_3\boldsymbol{e}_3$

   (b) $\boldsymbol{u} = x_1^2\boldsymbol{e}_1 + 2x_1x_2\boldsymbol{e}_2 + x_3^3\boldsymbol{e}_3$

   (c) $\boldsymbol{u} = x_2^2\boldsymbol{e}_1 + 2x_2x_3\boldsymbol{e}_2 + 4x_1^2\boldsymbol{e}_3$

3. The displacement components (u_1, u_2, u_3) are given by the relations $u_1 = x_1 - 2x_2$, $u_2 = 3x_1 + 2x_2$, $u_3 = 5x_3$. Verify that this displacement vector is continuously possible for a continuously deformed body. Determine the principal strains. Determine the principal axes of strain in the undeformed medium and in the deformed medium.

4. A one-dimensional problem of a prismatic bar (see the following figure) loaded under its own weight can be modeled by the stress field $\sigma_x = \sigma_x(x)$, $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$, with body forces $F_x = \rho g$, $F_y = F_z = 0$, where ρ is the mass density and g is the local acceleration of gravity. Using the equilibrium equations, show that the nonzero stress will be given by $\sigma_x = \rho g(l - x)$, where l is the length of the bar.



5. For the following state of stress, determine the principal stresses and directions and find the traction vector on a plane with unit normal $\boldsymbol{n} = (0, 1, 1)/\sqrt{2}$.

$$\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

(1) $\phi = x^2 - y^2 \qquad \underset{\sim}{u} = 2x\underset{\sim}{e}_1 + 3yz\underset{\sim}{e}_2 + xy\underset{\sim}{e}_3$

$$\underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{e}_1 + \frac{\partial\phi}{\partial y}\underset{\sim}{e}_2 = 2x\underset{\sim}{e}_1 - 2y\underset{\sim}{e}_2$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2 - 2 = 0$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 2 + 3z + 0 = 2 + 3z$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = \left(\frac{\partial}{\partial x}\underset{\sim}{e}_1 + \frac{\partial}{\partial y}\underset{\sim}{e}_2 + \frac{\partial}{\partial z}\underset{\sim}{e}_3\right)\left(u_x\underset{\sim}{e}_1 + u_y\underset{\sim}{e}_2 + u_z\underset{\sim}{e}_3\right) =$$

$$= \frac{\partial u_x}{\partial x}\underset{\sim}{e}_1\underset{\sim}{e}_1 + \frac{\partial u_y}{\partial x}\underset{\sim}{e}_1\underset{\sim}{e}_2 + \frac{\partial u_z}{\partial x}\underset{\sim}{e}_1\underset{\sim}{e}_3 +$$

$$+ \frac{\partial u_x}{\partial y}\underset{\sim}{e}_2\underset{\sim}{e}_1 + \frac{\partial u_y}{\partial y}\underset{\sim}{e}_2\underset{\sim}{e}_2 + \frac{\partial u_z}{\partial y}\underset{\sim}{e}_2\underset{\sim}{e}_3 +$$

$$+ \frac{\partial u_x}{\partial z}\underset{\sim}{e}_3\underset{\sim}{e}_1 + \frac{\partial u_y}{\partial z}\underset{\sim}{e}_3\underset{\sim}{e}_2 + \frac{\partial u_z}{\partial z}\underset{\sim}{e}_3\underset{\sim}{e}_3$$

$$= 2\underset{\sim}{e}_1\underset{\sim}{e}_1 + y\underset{\sim}{e}_1\underset{\sim}{e}_3 + 3z\underset{\sim}{e}_2\underset{\sim}{e}_2 + x\underset{\sim}{e}_2\underset{\sim}{e}_3 +$$

$$+ 3y\underset{\sim}{e}_3\underset{\sim}{e}_2$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 3yz & xy \end{vmatrix} = \underset{\sim}{e}_1(x - 3y) - \underset{\sim}{e}_2(y - 0) +$$

$$+ \underset{\sim}{e}_3 \cdot 0 =$$

$$= (x - 3y)\underset{\sim}{e}_1 - y\underset{\sim}{e}_2$$

(2) In general,

$$\underset{\sim}{u} = u_1 \underset{\sim}{e}_1 + u_2 \underset{\sim}{e}_2 + u_3 \underset{\sim}{e}_3$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \underset{\sim}{e}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) - \underset{\sim}{e}_2 \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right) + \underset{\sim}{e}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

$$\nabla^2 \underset{\sim}{u} = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} = \left(\frac{\partial}{\partial x_i} \underset{\sim}{e}_i \right) \cdot \left(\frac{\partial u_k}{\partial x_j} \underset{\sim}{e}_j \underset{\sim}{e}_k \right) =$$

$$= \frac{\partial^2 u_k}{\partial x_i \partial x_j} \delta_{ij} \underset{\sim}{e}_k = \frac{\partial^2 u_k}{\partial x_i \partial x_i} \underset{\sim}{e}_k =$$

$$= \frac{\partial^2 u_1}{\partial x_i \partial x_i} \underset{\sim}{e}_1 + \frac{\partial^2 u_2}{\partial x_i \partial x_i} \underset{\sim}{e}_2 + \frac{\partial^2 u_3}{\partial x_i \partial x_i} \underset{\sim}{e}_3 =$$

$$= \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) \underset{\sim}{e}_1 + \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} \right) \underset{\sim}{e}_2 +$$

$$+ \left(\frac{\partial^2 u_3}{\partial u_1^2} + \frac{\partial^2 u_3}{\partial u_2^2} + \frac{\partial^2 u_3}{\partial u_3^2} \right) \underset{\sim}{e}_3$$

$$\underset{\sim}{\nabla} \underset{\sim}{u} = \frac{\partial u_k}{\partial x_j} \underset{\sim}{e}_k \underset{\sim}{e}_j = \frac{\partial u_1}{\partial x_1} \underset{\sim}{e}_1 \underset{\sim}{e}_1 + \frac{\partial u_1}{\partial x_2} \underset{\sim}{e}_1 \underset{\sim}{e}_2 + \frac{\partial u_1}{\partial x_3} \underset{\sim}{e}_1 \underset{\sim}{e}_3 +$$

$$+ \frac{\partial u_2}{\partial x_1} \underset{\sim}{e}_2 \underset{\sim}{e}_1 + \frac{\partial u_2}{\partial x_2} \underset{\sim}{e}_2 \underset{\sim}{e}_2 + \frac{\partial u_2}{\partial x_3} \underset{\sim}{e}_2 \underset{\sim}{e}_3 + \frac{\partial u_3}{\partial x_1} \underset{\sim}{e}_3 \underset{\sim}{e}_1 +$$

$$+ \frac{\partial u_3}{\partial x_2} \underset{\sim}{e}_3 \underset{\sim}{e}_2 + \frac{\partial u_3}{\partial x_3} \underset{\sim}{e}_3 \underset{\sim}{e}_3$$

$$tr(\underset{\sim}{\nabla}\underset{\sim}{u}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \underset{\sim}{\nabla}\cdot\underset{\sim}{u}$$

For each of the displacement fields:

(a) $\underset{\sim}{u} = x_1 \underset{\sim}{e}_1 + x_1 x_2 \underset{\sim}{e}_2 + 2x_1 x_2 x_3 \underset{\sim}{e}_3$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 1 + x_1 + 2x_1x_2$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{u} = \underset{\sim}{e}_1(2x_1x_3 - 0) - \underset{\sim}{e}_2(2x_2x_3 - 0) + \underset{\sim}{e}_3(x_2 - 0) =$$

$$= 2x_1x_3\underset{\sim}{e}_1 - 2x_2x_3\underset{\sim}{e}_2 + x_2\underset{\sim}{e}_3$$

$$\nabla^2\underset{\sim}{u} = 0$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = \underset{\sim}{e}_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2\underset{\sim}{e}_1 + x_1\underset{\sim}{e}_2\underset{\sim}{e}_2 + 2x_2x_3\underset{\sim}{e}_3\underset{\sim}{e}_1 + 2x_1x_3\underset{\sim}{e}_3\underset{\sim}{e}_2 +$$

$$+2x_1x_2\underset{\sim}{e}_3\underset{\sim}{e}_3$$

$$tr(\underset{\sim}{\nabla}\underset{\sim}{u}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 1 + x_1 + 2x_1x_2$$

(b) $\underset{\sim}{u} = x_1^2\underset{\sim}{e}_1 + 2x_1x_2\underset{\sim}{e}_2 + x_3^3\underset{\sim}{e}_3$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 2x_1 + 2x_1 + 3x_2^2 = 4x_1 + 3x_2^2$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{u} = \underset{\sim}{e}_1(0-0) - \underset{\sim}{e}_2(0-0) + \underset{\sim}{e}_3(2x_2 - 0) = 2x_2\underset{\sim}{e}_3$$

$$\nabla^2\underset{\sim}{u} = \underset{\sim}{e}_1(2) + \underset{\sim}{e}_2(0) + \underset{\sim}{e}_3(6x_3) = 2\underset{\sim}{e}_1 + 6x_3\underset{\sim}{e}_3$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = 2x_1\underset{\sim}{e}_1\underset{\sim}{e}_1 + 0\cdot\underset{\sim}{e}_1\underset{\sim}{e}_2 + 0\cdot\underset{\sim}{e}_1\underset{\sim}{e}_3 +$$

$$+ 2x_2\underset{\sim}{e}_2\underset{\sim}{e}_1 + 2x_1\underset{\sim}{e}_2\underset{\sim}{e}_2 + 0\cdot\underset{\sim}{e}_2\underset{\sim}{e}_3 +$$

$$+ 0\cdot\underset{\sim}{e}_3\underset{\sim}{e}_1 + 0\cdot\underset{\sim}{e}_3\underset{\sim}{e}_2 + 3x_2^2\underset{\sim}{e}_3\underset{\sim}{e}_2 =$$

$$= 2x_1\underset{\sim}{e}_1\underset{\sim}{e}_1 + 2x_2\underset{\sim}{e}_2\underset{\sim}{e}_1 + 2x_1\underset{\sim}{e}_2\underset{\sim}{e}_2 + 3x_2^2\underset{\sim}{e}_3\underset{\sim}{e}_2$$

$$\mathrm{tr}(\underset{\sim}{\nabla}\underset{\sim}{u}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 4x_1 + 3x_2^2$$

(c) $\underset{\sim}{u} = x_2^2\underset{\sim}{e}_1 + 2x_2x_3\underset{\sim}{e}_2 + 4x_1^2\underset{\sim}{e}_3$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 2x_3$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{u} = \underset{\sim}{e}_1(0 - 2x_2) - \underset{\sim}{e}_2(8x_1 - 0) + \underset{\sim}{e}_3(0 - 2x_2)$$

$$= -2x_2\underset{\sim}{e}_1 - 8x_1\underset{\sim}{e}_2 - 2x_2\underset{\sim}{e}_3$$

$$\nabla^2\underset{\sim}{u} = \underset{\sim}{e}_1(2) + \underset{\sim}{e}_3(8) = 2\underset{\sim}{e}_1 + 8\underset{\sim}{e}_3$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = 2x_2\underset{\sim}{e}_1\underset{\sim}{e}_2 + 2x_3\underset{\sim}{e}_2\underset{\sim}{e}_2 + 2x_2\underset{\sim}{e}_2\underset{\sim}{e}_3 + 8x_1\underset{\sim}{e}_3\underset{\sim}{e}_1$$

$$\mathrm{tr}(\underset{\sim}{\nabla}\underset{\sim}{u}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 2x_3$$

③ $u_1 = x_1 - 2x_2 \qquad u_2 = 3x_1 + 2x_2 \qquad u_3 = 5x_3$

$$J = \det\left(\underset{\sim}{I} + \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}\right) =$$

$$= \begin{vmatrix} 1+1 & -2 & 0 \\ 3 & 1+2 & 0 \\ 0 & 0 & 1+5 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 6(6+6) = 72$$

$J > 0 \Rightarrow$ deformation admissible

$$2\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j} \Rightarrow$$

$$\Rightarrow 2\varepsilon_{11} = 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2 = 2 + 1^2 + 3^2 + 0 = 12 \Rightarrow$$

$$\Rightarrow \varepsilon_{11} = 6$$

Similarly

$$\varepsilon_{22} = 6 \quad \varepsilon_{33} = \frac{35}{2} \quad \varepsilon_{12} = \frac{5}{2} \quad \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\underset{\sim}{\varepsilon} = \begin{vmatrix} 6 & \frac{5}{2} & 0 \\ \frac{5}{2} & 6 & 0 \\ 0 & 0 & \frac{35}{2} \end{vmatrix}$$

$$\det(\underset{\sim}{\varepsilon} - \lambda \underset{\sim}{I}) = 0 \iff \begin{vmatrix} 6-\lambda & \frac{5}{2} & 0 \\ \frac{5}{2} & 6-\lambda & 0 \\ 0 & 0 & \frac{35}{2}-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left(\frac{35}{2} - \lambda\right)\left[(6-\lambda)^2 - \frac{25}{4}\right] = 0 \iff \left(\frac{35}{2} - \lambda\right)\left(6 - \lambda - \frac{5}{2}\right)\left(6 - \lambda + \frac{5}{2}\right) = 0$$

$$\Rightarrow \quad (17.5-\lambda)(3.5-\lambda)(8.5-\lambda)=0 \quad \Rightarrow$$

$$\Rightarrow \quad \lambda_1 = 17.5 \quad \lambda_2 = 8.5 \quad \lambda_3 = 3.5$$

Principal directions

$\underline{\lambda_1 = 17.5}$

$$\begin{bmatrix} 6-17.5 & \frac{5}{2} & 0 \\ \frac{5}{2} & 6-17.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_1^{(1)} \\ N_2^{(1)} \\ N_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow$$

Also $\quad N_1^{(1)} + N_2^{(1)} + N_3^{(1)} = 0$

$$\Rightarrow \begin{cases} -11.5\, N_1^{(1)} + 2.5\, N_2^{(1)} + 0 = 0 \\ 2.5\, N_1^{(1)} - 11.5\, N_2^{(1)} + 0 = 0 \\ 0\, N_3^{(1)} = 0 \\ N_1^{(1)2} + N_2^{(1)2} + N_3^{(1)2} = 0 \end{cases} \quad \Rightarrow \quad \begin{array}{l} N_1^{(1)} = N_2^{(1)} = 0 \\ N_3^{(1)} = 1 \end{array}$$

$\lambda_2 = 8.5$

$$\begin{bmatrix} 6-8.5 & \frac{5}{2} & 0 \\ \frac{5}{2} & 6-8.5 & 0 \\ 0 & 0 & \frac{35}{2}-8.5 \end{bmatrix} \begin{bmatrix} N_1^{(2)} \\ N_2^{(2)} \\ N_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow$$

$$N_1^{(2)} + N_2^{(2)} + N_3^{(2)} = 1$$

$$\begin{cases} -2.5N_1^{(2)} + 2.5N_2^{(2)} = 0 \\ 2.5N_1^{(2)} - 2.5N_2^{(2)} = 0 \\ 9N_3^{(2)} = 0 \\ {N_1^{(2)}}^2 + {N_2^{(2)}}^2 + {N_3^{(2)}}^2 = 1 \end{cases} \Rightarrow \quad N_1^{(2)} = N_2^{(2)} = \pm\frac{1}{\sqrt{2}}, \quad N_3^{(2)} = 0$$

$\underline{\lambda_3 = 3.5}$

$$\begin{bmatrix} 6-3.5 & \frac{5}{2} & 0 \\ \frac{5}{2} & 6-3.5 & 0 \\ 0 & 0 & \frac{35}{2}-3.5 \end{bmatrix} \begin{bmatrix} N_1^{(3)} \\ N_2^{(3)} \\ N_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$${N_1^{(3)}}^2 + {N_2^{(3)}}^2 + {N_3^{(3)}}^2 = 1$$

$$\Rightarrow \begin{cases} 2.5N_1^{(3)} + 2.5N_2^{(3)} = 0 \\ 2.5N_1^{(3)} + 2.5N_2^{(3)} = 0 \\ 14N_3^{(3)} = 0 \\ {N_1^{(3)}}^2 + {N_2^{(3)}}^2 + {N_3^{(3)}}^2 = 1 \end{cases} \Rightarrow \quad N_1^{(3)} = \pm\frac{1}{\sqrt{2}}, \quad N_2^{(3)} = \mp\frac{1}{\sqrt{2}}, \quad N_3^{(3)} = 0$$

Principal directions in the deformed medium:

$$n_\alpha^{(i)} \sqrt{1+2MF_i} = \left(\delta_{\alpha\beta} + u_{\alpha,\beta}\right) N_\beta^{(i)}$$

But $MF_i = \lambda_i \qquad i=1,2,3$

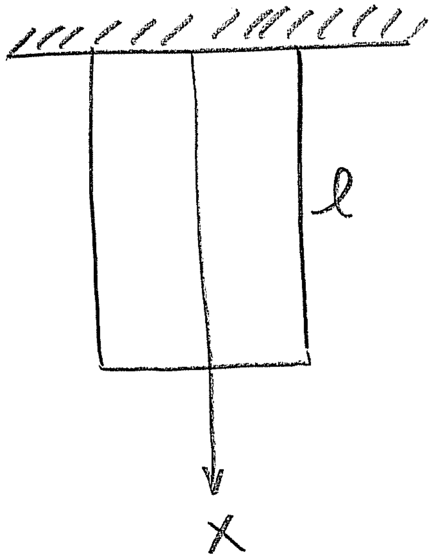
Then

$$n_1^{(1)} = n_2^{(1)} = 0 \qquad n_3^{(1)} = \pm 1$$

$$n_1^{(2)} = n_3^{(2)} = 0 \qquad n_2^{(2)} = \pm 1$$

$$n_2^{(3)} = n_3^{(3)} = 0 \qquad n_1^{(3)} = \pm 1$$

(4)



$\sigma_x = \sigma_x(x)$

$\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$

body forces

$F_x = \rho g \quad F_y = F_z = 0$

Equilibrium equations:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + F_x = 0 \Rightarrow \dfrac{\partial \sigma_x}{\partial x} + F_x = 0 & (1) \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + F_y = 0 \Rightarrow 0 = 0 & (2) \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + F_z = 0 \Rightarrow 0 = 0 & (3) \end{cases}$$

$$(1) \Rightarrow \frac{\partial \sigma_x}{\partial x} + \rho g = 0 \Rightarrow \sigma_x = -\rho g x + C$$

Obviously, σ_x is not a function of y or z.
The ~~constant~~ C is obtained from the boundary condition.

For $x = l \quad \sigma_x = 0 \Rightarrow -\rho g l + C = 0 \Rightarrow C = \rho g l \Rightarrow$

$$\Rightarrow \boxed{\sigma_x = -\rho g x + \rho g l = \rho g (l - x)}$$

(5)

$$\underset{\sim}{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\det(\underset{\sim}{\sigma} - \lambda \underset{\sim}{I}) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} =$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 1 & 2 \end{vmatrix} =$$

$$= (3-\lambda)(\lambda^2-4) - (-\lambda-2) + (2+\lambda) =$$

$$= 3\lambda^2 - \lambda^3 - 12 + 4\lambda + \lambda + 2 + \lambda + 2 =$$

$$= -\lambda^3 + 3\lambda^2 + 6\lambda - 8 = -(\lambda^3+8) + 3\lambda(\lambda+2)$$

But $\lambda^3+8 = \lambda^3 + 2^3$

Now, we can use for the above expression the following identity

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$\lambda^3+2^3 = (\lambda+2)(\lambda^2-2\lambda+4)$$

Thus,

$$\det(\underset{\sim}{\sigma} - \lambda\underset{\sim}{I}) = -(\lambda+2)(\lambda^2-2\lambda+4) + 3\lambda(\lambda+2) =$$

$$= -(\lambda+2)(\lambda^2-2\lambda+4-3\lambda) =$$

$$= -(\lambda+2)(\lambda^2-5\lambda+4) =$$

$$= -(\lambda+2)(\lambda-1)(\lambda-4)$$

$$\det(\underset{\sim}{\sigma} - \lambda \underset{\sim}{I}) = 0 \iff (\lambda+2)(\lambda-1)(\lambda-4) = 0 \implies$$

$$\implies \lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = -2$$

Principal stresses $\sigma_1 = 4 \quad \sigma_2 = 1 \quad \sigma_3 = -2$

Principal directions :

$\underline{\lambda_1 = 4}$

$$\begin{bmatrix} 3-4 & 1 & 1 \\ 1 & -4 & 2 \\ 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} N_1^{(1)} \\ N_2^{(1)} \\ N_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$$

$$\implies \begin{cases} -N_1^{(1)} + N_2^{(1)} + N_3^{(1)} = 0 & (1) \\ N_1^{(1)} - 4N_2^{(1)} + 2N_3^{(1)} = 0 & (2) \\ N_1^{(1)} + 2N_2^{(1)} - 4N_3^{(1)} = 0 & (3) \end{cases} \implies (2)-(3) \iff$$

$$-6N_2^{(1)} + 6N_3^{(1)} = 0 \implies N_2^{(1)} = N_3^{(1)}$$

$$(1) \implies N_1^{(1)} = N_2^{(1)} + N_3^{(1)} = 2N_2^{(1)}$$

But $N_1^{(1)^2} + N_2^{(1)^2} + N_3^{(1)^2} = 1 \implies \left[2N_2^{(1)}\right]^2 + N_2^{(1)^2} + N_2^{(1)^2} = 1 \implies$

$$\implies 6N_2^{(1)^2} = 1 \implies \underline{N_2^{(1)} = N_3^{(1)} = \pm\frac{1}{\sqrt{6}} \quad N_1^{(1)} = \pm\frac{2}{\sqrt{6}}}$$

$\underline{\lambda_2 = 1}$

$$\begin{bmatrix} 3-1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} N_1^{(2)} \\ N_2^{(2)} \\ N_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2N_1^{(2)} + N_2^{(2)} + N_3^{(2)} = 0 & (1) \\ N_1^{(2)} - N_2^{(2)} + 2N_3^{(2)} = 0 & (2) \\ N_1^{(2)} + 2N_2^{(2)} - N_3^{(2)} = 0 & (3) \end{cases}$$

$$(2) - (3) \Leftrightarrow \quad -3N_2^{(2)} + 3N_3^{(2)} = 0 \quad \Rightarrow \quad N_2^{(2)} = N_3^{(2)}$$

$$(1) \quad \Rightarrow \quad 2N_1^{(2)} + 2N_2^{(2)} = 0 \Rightarrow \quad N_1^{(2)} = -N_2^{(2)}$$

But
$$N_1^{(2)^2} + N_2^{(2)^2} + N_3^{(2)^2} = 1 \Rightarrow 3N_2^{(2)^2} = 1 \Rightarrow \underline{N_2^{(2)} = \pm\frac{1}{\sqrt{3}}} = N_3^{(2)}$$

$$\underline{N_1^{(2)} = \mp\frac{1}{\sqrt{3}}}$$

$\lambda_3 = -2$

$$\begin{bmatrix} 3+2 & 1 & 1 \\ 1 & +2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} N_1^{(3)} \\ N_2^{(3)} \\ N_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 5N_1^{(3)} + N_2^{(3)} + N_3^{(3)} = 0 & (1) \\ N_1^{(3)} + 2N_2^{(3)} + 2N_3^{(3)} = 0 & (2) \\ N_1^{(3)} + 2N_2^{(3)} + 2N_3^{(3)} = 0 & (3) \end{cases}$$

$$(1) - 5(2) \Leftrightarrow \quad -9N_2^{(3)} - 9N_3^{(3)} = 0 \Rightarrow N_2^{(3)} = -N_3^{(3)}$$

$$(1) \Rightarrow 5N_1^{(3)} = 0 \Rightarrow N_1^{(3)} = 0$$

But,

$$N_1^{(3)^2} + N_2^{(3)^2} + N_3^{(3)^2} = 1 \Rightarrow N_2^{(3)^2} + N_2^{(3)^2} = 1 \Rightarrow N_2^{(3)} = \pm \frac{1}{\sqrt{2}}$$

$$\underline{N_1^{(3)} = 0 \quad N_2^{(3)} = \pm \frac{1}{\sqrt{2}} \quad N_3^{(3)} = \mp \frac{1}{\sqrt{2}}}$$

$$\underset{\sim}{t} = \underset{\sim}{\sigma}^T \underset{\sim}{n} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$