ME 548 Elasticity, Fall 2013
Exam 2
Assigned: Thursday, Dec. 5, 2013
Due: Monday, Dec. 9, 2013 by 7:00 pm PST

1) Show that, if there are no body forces, the dilatation e must satisfy the condition

$$\nabla^2 e = 0 .$$

2) In the absence of body forces, show that the following stresses

$$\sigma_x = kxy,\ \sigma_y = kx,\ \sigma_z = \nu kx(1+y)$$
$$\tau_{xy} = -\frac{1}{2}ky^2,\ \tau_{xz} = \tau_{yz} = 0,\ k = constant$$

satisfy the plane strain stress formulation relations.

3) If a material is incompressible ($\nu = 0.5$), a state of hydrostatic stress $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ produces no strain. One way to write the corresponding stress-strain relations is

$$\sigma_{ij} = 2\mu e_{ij} - q\delta_{ij} ,$$

where q is an unknown hydrostatic pressure which will generally vary with position. Also, the condition of incompressibility requires that the dilatation

$$e \equiv e_{kk} = 0 .$$

Show that the stress components and the hydrostatic pressure q must satisfy the equations

$$\nabla^2 q = \operatorname{div} \boldsymbol{p} \quad \text{and} \quad \sigma_{xx} + \sigma_{yy} = -2q ,$$

where $\boldsymbol{p}$ is the body force.

4) Derive results to Eqs. (4-7.1) through (4-7.6) from the textbook for the case of hydrostatic compression $\sigma_{11} = \sigma_{11} = \sigma_{11} = -p$ and $\sigma_{12} = \sigma_{23} = \sigma_{13} = 0$, where p is pressure.

5) Derive Equation (4-11.9) from the textbook.

6) Derive Equation (d-1) on page 300 of the textbook.

(1) $\sigma_{ij} = \lambda e \delta_{ij} + 2\mu \varepsilon_{ij}$

$\sigma_x = \lambda e + 2\mu \varepsilon_x$

$\sigma_y = \lambda e + 2\mu \varepsilon_y$

$\sigma_z = \lambda e + 2\mu \varepsilon_z$

$\tau_{xy} = 2\mu \varepsilon_{xy}$

$\tau_{yz} = 2\mu \varepsilon_{yz}$

$\tau_{zx} = 2\mu \varepsilon_{zx}$

Equilibrium :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + P_x = 0 \qquad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + P_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + P_z = 0$$

$$(1) \Rightarrow \lambda \frac{\partial e}{\partial x} + 2\mu \left(\frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_{xz}}{\partial z} \right) + P_x = 0$$

But $\dfrac{\partial \varepsilon_x}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u_x}{\partial x} \right) = \dfrac{\partial^2 u_x}{\partial x^2}$

$$\frac{\partial \varepsilon_{xy}}{\partial y} = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = \frac{1}{2}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} \right)$$

$$\frac{\partial \varepsilon_{xz}}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right) \Rightarrow$$

$$\Rightarrow 2\left(\frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_{xz}}{\partial z}\right) = 2\frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) +$$

$$+ \left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right) = \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \left(\frac{\partial^2 u_x}{\partial x^2} +\right.$$

$$\left. + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right) = \nabla^2 u_x + \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) =$$

$$= \nabla^2 u_x + \frac{\partial e}{\partial x}$$

Thus,

$$(1) \Leftrightarrow \lambda \frac{\partial e}{\partial x} + \mu\left(\nabla^2 u_x + \frac{\partial e}{\partial x}\right) + P_x = 0 \Leftrightarrow$$

$$\Leftrightarrow (\lambda+\mu)\frac{\partial e}{\partial x} + \mu \nabla^2 u_x + P_x = 0 \qquad (2)$$

Also, similarly it can be shown that:

$$(\lambda+\mu)\frac{\partial e}{\partial y} + \mu \nabla^2 u_y + P_y = 0 \qquad (3)$$

$$(\lambda+\mu)\frac{\partial e}{\partial z} + \mu \nabla^2 u_z + P_z = 0 \qquad (4)$$

Adding (2) + (3) + (4), after multiplying by $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ respectively:

$$(\lambda+\mu)\left[\frac{\partial}{\partial x}\operatorname{div}\underset{\sim}{u}\,\underset{\sim}{i} + \frac{\partial}{\partial y}\operatorname{div}\underset{\sim}{u}\,\underset{\sim}{j} + \frac{\partial}{\partial z}\operatorname{div}\underset{\sim}{u}\,\underset{\sim}{k}\right] + \mu \underset{\sim}{\nabla}^2 \underset{\sim}{u} + \underset{\sim}{P} = \underset{\sim}{0}$$

OR,

$$\boxed{(\lambda+\mu)\,\underset{\sim}{\nabla}(\operatorname{div}\underset{\sim}{u}) + \mu\,\nabla^2\underset{\sim}{u} + \underset{\sim}{p} = 0}$$

If $\underset{\sim}{p} = 0 \Rightarrow$

$$(\lambda+\mu)\,\underset{\sim}{\nabla}(\operatorname{div}\underset{\sim}{u}) + \mu\,\nabla^2\underset{\sim}{u} = 0 \quad (5)$$

Applying $\underset{\sim}{\nabla}$ to (5) :

$$(\lambda+\mu)\,\nabla^2\operatorname{div}\underset{\sim}{u} + \mu\,\operatorname{div}\nabla^2\underset{\sim}{u} = 0 \quad (6)$$

But $\operatorname{div}\underset{\sim}{\nabla} = \nabla^2$

$\operatorname{div}\nabla^2 = \nabla^2\operatorname{div}$

Then (6) $\Leftrightarrow (\lambda+\mu)\operatorname{div}\nabla^2\underset{\sim}{u} + \mu\operatorname{div}\nabla^2\underset{\sim}{u} = 0 \Leftrightarrow$

$\Leftrightarrow \quad (\lambda+2\mu)\operatorname{div}\nabla^2\underset{\sim}{u} = 0 \Leftrightarrow (\lambda+2\mu)\,\nabla^2\operatorname{div}\underset{\sim}{u} = 0 \Rightarrow$

$\Rightarrow \quad \nabla^2\operatorname{div}\underset{\sim}{u} = 0 \Leftrightarrow \boxed{\nabla^2\underset{\sim}{e} = 0}$

because $\operatorname{div}\underset{\sim}{u} = \underset{\sim}{e}$

(2) $\sigma_x = kxy \quad \sigma_y = kx \quad \sigma_z = \nu kx(1+y)$

$\tau_{xy} = -\frac{1}{2}ky^2 \quad \tau_{xz} = \tau_{yz} = 0 \quad k = \text{constant.}$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{1}{E}\left(kxy - \nu\left(kx + \nu kx + \nu kxy\right)\right)$$

$$= \frac{1}{E}\left[kxy - \nu kx - \nu^2 kx - \nu^2 kxy\right] =$$

$$= \frac{1}{E}\left[kxy(1-\nu^2) - \nu kx(1+\nu)\right] =$$

$$= \frac{1}{E}\left[kxy(1-\nu)(1+\nu) - \nu kx(1+\nu)\right] =$$

$$= \frac{1}{E}\left\{kx(1+\nu)\left[y(1-\nu) - k\right]\right\} =$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] = \frac{1}{E}\left[kx - \nu\left(kxy + \nu kx + \nu kxy\right)\right]$$

$$= \frac{1}{E}\left[kx - \nu kxy - \nu^2 kx - \nu^2 kxy\right]$$

$$= \frac{1}{E}\left[kx(1-\nu^2) - \nu kxy(1+\nu)\right] =$$

$$= \frac{1}{E}\left[kx(1-\nu)(1+\nu) - \nu kxy(1+\nu)\right] =$$

$$= \frac{1}{E}\left[kx(1+\nu)(1-\nu-\nu y)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x+\sigma_y)\right] = \frac{1}{E}\left[\nu kx(1+y) - \nu(kxy+kx)\right]$$

$$= \frac{1}{E}\left[\nu kx + \nu kxy - \nu kxy - \nu kx\right] = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\frac{\tau_{xy}}{G} = \frac{1}{2}\frac{\tau_{xy}}{\frac{E}{2(1+\nu)}} = \frac{(1+\nu)\,\tau_{xy}}{E} = -\frac{(1+\nu)ky^2}{2E}$$

$$\varepsilon_{xz} = \frac{(1+\nu)\,\tau_{xz}}{E} = 0$$

$$\varepsilon_{yz} = \frac{(1+\nu)\,\tau_{yz}}{E} = 0$$

Thus,

$\varepsilon_x \neq 0 \quad \varepsilon_y \neq 0 \quad \varepsilon_z = 0$

$\gamma_{xy} = 2\varepsilon_{xy} \neq 0 \quad \gamma_{xz} = \gamma_{yz} = 0$

| 2D plane stress

Now, let's verify that the stresses are compatible:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \iff ky - ky + 0 = 0 \checkmark$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \iff 0 + 0 + 0 = 0 \checkmark$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \iff 0 + 0 + 0 = 0 \checkmark$$

③ $\nu = 0.5$

$$\sigma_x = \sigma_y = \sigma_z$$

$$\sigma_{ij} = 2\mu \varepsilon_{ij} - q\,\delta_{ij}$$

$$e = \varepsilon_{kk} = 0$$

Show that $\nabla^2 q = \operatorname{div} \underset{\sim}{p}$ and $\sigma_x + \sigma_y = -2q$

$\underset{\sim}{p}$ = body force.

<u>Proof</u> :

$$\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + p_x = 0 \quad \Big| \text{ Apply } \frac{\partial}{\partial x} \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + p_y = 0 \quad \Big| \text{ Apply } \frac{\partial}{\partial y} \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + p_z = 0 \quad \Big| \text{ Apply } \frac{\partial}{\partial z}
\end{aligned}\right\} \Rightarrow$$

$$\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right) + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + 2\frac{\partial^2 \tau_{xz}}{\partial x \partial z} + 2\frac{\partial^2 \tau_{yz}}{\partial y \partial z} + \left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z}\right) = 0 \qquad (1)$$

$$\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z} = \operatorname{div} \underset{\sim}{p} = \underset{\sim}{\nabla} \cdot \underset{\sim}{p}$$

But, from the problem statement:

$$\sigma_x = 2\mu \varepsilon_x - 2 \qquad \tau_{xy} = 2\mu \varepsilon_{xy}$$

$$\sigma_y = 2\mu \varepsilon_y - 2 \qquad \tau_{yz} = 2\mu \varepsilon_{yz}$$

$$\sigma_z = 2\mu \varepsilon_z - 2 \qquad \tau_{xz} = 2\mu \varepsilon_{xz}$$

Then

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2} = 2\mu \left(\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial y^2} + \frac{\partial^2 \varepsilon_z}{\partial z^2} \right) - \nabla^2 2$$

And

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + 2\frac{\partial^2 \tau_{xz}}{\partial x \partial z} + 2\frac{\partial^2 \tau_{yz}}{\partial y \partial z} = 2\mu \left(2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} + 2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} + 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \right)$$

Thus,

$$(1) \Leftrightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{p} + 2\mu \left(\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial y^2} + \frac{\partial^2 \varepsilon_z}{\partial z^2} \right) - \nabla^2 2 + 2\mu \left(2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} + 2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} + 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \right) = 0$$

Applying the strain compatibility equations, the last bracket can be replaced as follows:

$$(1) \iff \underset{\sim}{\nabla}\cdot \underset{\sim}{p} + 2\mu\left(\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial y^2} + \frac{\partial^2 \varepsilon_z}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} + \right.$$

$$\left. + \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}\right) - \nabla^2 q = 0$$

OR,

$$\underset{\sim}{\nabla}\cdot \underset{\sim}{p} + 2\mu \nabla^2 e - \nabla^2 q = 0$$

But $\nabla^2 e = 0 \Rightarrow \underset{\sim}{\nabla}\cdot \underset{\sim}{p} - \nabla^2 q = 0$

$$\boxed{\underset{\sim}{\nabla}\cdot \underset{\sim}{p} = \nabla^2 q}$$

Also,

$$\sigma_x + \sigma_y + \sigma_z = 2\mu(\varepsilon_x + \varepsilon_y + \varepsilon_z) - 3q$$

$$= 2\mu e - 3q = -3q$$

But $\sigma_x = \sigma_y = \sigma_z \Rightarrow \sigma_z = -q \Rightarrow$

$$\Rightarrow \sigma_x + \sigma_y = -3q - \sigma_z = -3q + q = -2q$$

$$\boxed{\sigma_x + \sigma_y = -2q}$$

④ Hydrostatic pressure :

$$\sigma_x = \sigma_y = \sigma_z = -p \qquad \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$$

$$\left.\begin{array}{llll} (1) & \sigma_x = \lambda e + 2G\varepsilon_x = -p & \varepsilon_{xy} = 0 & (4) \\ (2) & \sigma_y = \lambda e + 2G\varepsilon_y = -p & \varepsilon_{xz} = 0 & (5) \\ (3) & \sigma_z = \lambda e + 2G\varepsilon_z = -p & \varepsilon_{yz} = 0 & (6) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad (1)+(2)+(3) \Rightarrow \quad 3\lambda e + 2Ge = -3p \Rightarrow$$

$$\Rightarrow (3\lambda + 2G)e = -3p \Rightarrow \quad e = -\frac{3p}{3\lambda + 2G}$$

$$(1) \Rightarrow \varepsilon_x = \frac{1}{2G}\left[-p - \lambda e\right] = \frac{1}{2G}\left[-p - \frac{3\lambda p}{3\lambda + 2G}\right] = \frac{1}{2G}\left[\frac{-6\lambda p - 2Gp}{3\lambda + 2G}\right] =$$

$$= -\frac{(3\lambda + G)p}{G(3\lambda + 2G)}$$

$$\begin{cases} \varepsilon_x = \varepsilon_y = \varepsilon_z = -\dfrac{(3\lambda + G)p}{G(3\lambda + 2G)} \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \end{cases}$$

Integrating the strains :

$$u_x = -\frac{(3\lambda + G)p}{G(3\lambda + 2G)}x \qquad u_y = -\frac{(3\lambda + G)p}{G(3\lambda + 2G)}y \quad u_z = -\frac{(3\lambda + G)p}{G(3\lambda + 2G)}z$$

The elastic constantants as defined in (4.-7.2) - (4.-7.5) still hold true.

(5)

According to (4-11.8)

$$U = \left(\frac{1}{2}\lambda + G\right) J_1^2 - 2G J_2 - c J_1 T + \frac{3}{2} cKT^2$$

$$J_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$J_2 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2$$

Then,

$$U = \left(\frac{1}{2}\lambda + G\right)\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)^2 - 2G\left(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33} - \right.$$
$$\left. - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2\right) - c\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) T + \frac{3}{2} cKT^2$$

$$= \frac{1}{2}\lambda\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)^2 + G\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\varepsilon_{11}\varepsilon_{22} + 2\varepsilon_{11}\varepsilon_{33} + \right.$$
$$\left. + 2\varepsilon_{11}\varepsilon_{33}\right) - 2G\left(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2\right) -$$
$$- c\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + \frac{3}{2} cKT^2$$

$$= \frac{1}{2}\lambda\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)^2 + G\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2\right) -$$
$$- c\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + \frac{3}{2} cKT^2$$

✓

(6)

$$
\begin{cases}
\varepsilon_x = \dfrac{1}{E}\left[(1+\nu)\sigma_x - \nu I_1\right] + KT \\
\varepsilon_y = \dfrac{1}{E}\left[(1+\nu)\sigma_y - \nu I_1\right] + KT \\
\varepsilon_z = \dfrac{1}{E}\left[(1+\nu)\sigma_z - \nu I_1\right] + KT \\
\gamma_{xy} = \dfrac{2(1+\nu)}{E}\tau_{xy} \\
\gamma_{xz} = \dfrac{2(1+\nu)}{E}\tau_{xz} \\
\gamma_{yz} = \dfrac{2(1+\nu)}{E}\tau_{yz}
\end{cases}
\qquad (1)
$$

$$
\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} \quad (2) \Rightarrow
$$

$\Rightarrow$ By substituting (1)-1 and (1)-2, (1)-3 into (2)

$$
\frac{1}{E}\left[(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2}\right) - \nu\left(\frac{\partial^2 I_1}{\partial z^2} + \frac{\partial^2 I_1}{\partial x^2}\right)\right] + K\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2}\right) =
$$

$$
= \frac{2(1+\nu)}{E}\,\frac{\partial^2 \tau_{xz}}{\partial x \partial z} \Rightarrow
$$

$$
\Rightarrow (1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2}\right) - \nu\left(\frac{\partial^2 I_1}{\partial z^2} + \frac{\partial^2 I_1}{\partial x^2}\right) = 2(1+\nu)\frac{\partial^2 \tau_{xz}}{\partial x \partial z} -
$$

$$
- K\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2}\right) \Rightarrow
$$

Equilibrium :

$$\left.\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} &\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} - B_x \\ &\frac{\partial \tau_{xz}}{\partial x} = -\frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \sigma_z}{\partial z} - B_z \end{aligned}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} &\frac{\partial^2 \tau_{xz}}{\partial x \partial z} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial x \partial y} - \frac{\partial B_x}{\partial x} \\ &\frac{\partial^2 \tau_{xz}}{\partial x \partial z} = -\frac{\partial^2 \tau_{yz}}{\partial y \partial z} - \frac{\partial^2 \sigma_z}{\partial z^2} - \frac{\partial B_z}{\partial z} \end{aligned}\right.$$

$$\Rightarrow 2(1+\nu)\frac{\partial^2 \tau_{xz}}{\partial x \partial z} = -2(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right) - 2(1+\nu)\left(\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{yz}}{\partial y \partial z}\right) - 2(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z}\right) \qquad (3)$$

But,

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{yz}}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}\right) =$$

$$= \frac{\partial}{\partial y}\left(-\frac{\partial \sigma_y}{\partial y} - B_y\right) = -\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial B_y}{\partial y}$$

Plugging this result into (3)

$$2(1+\nu)\frac{\partial^2 \tau_{xz}}{\partial x \partial z} = -2(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right) - 2(1+\nu)\left(-\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial B_y}{\partial y}\right) - 2(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z}\right)$$

The equilibrium equation becomes:

$$(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2}\right) - \nu\left(\frac{\partial^2 I_1}{\partial z^2} + \frac{\partial^2 I_1}{\partial x^2}\right) = -2(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right) - 2(1+\nu)\left(-\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial B_y}{\partial y}\right) - 2(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z}\right) - k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

OR,

$$(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2}\right) - \nu\left(\frac{\partial^2 I_1}{\partial z^2} + \frac{\partial^2 I_1}{\partial x^2}\right) + 2(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right) - 2(1+\nu)\frac{\partial^2 \sigma_y}{\partial y^2} = 2(1+\nu)\left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right) - k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

OR,

$$\nabla^2(I_1 + EkT) - \frac{\partial^2 (I_1 + EkT)}{\partial x^2} - (1+\nu)\nabla^2 \sigma_y = (1+\nu)\left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)$$