Final exam, ME 548 Elasticity

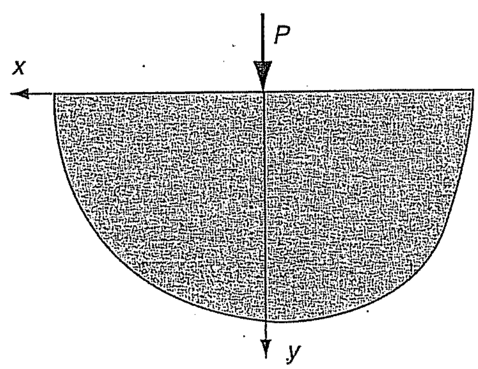
Due: Monday, 12/16/2013, 6pm PST

(each problem is 25 pts.)

1) Show that the octahedral normal and shear stresses can be expressed by

$$\sigma_{oct} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$
$$\tau_{oct} = \frac{1}{3}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 + 6\tau_{yz}^2 + 6\tau_{zx}^2]^{1/2}$$

2) The plane stress solution for a semi-infinite elastic solid under a concentrated point loading is given by



$$\sigma_x = -\frac{2Px^2y}{\pi(x^2+y^2)^2}$$
$$\sigma_y = -\frac{2Py^3}{\pi(x^2+y^2)^2}$$
$$\tau_{xy} = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}$$

Calculate the principal stresses and the maximum shear stress at any point in the body and plot contours of τ_{max}.

3) A *hydrostatic stress field* is specified by

$$\sigma_{ij} = -p\delta_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

where $p = p(x_1, x_2, x_3)$ and may be called the pressure. Show that the equilibrium equations imply that the pressure must satisfy the relation $\nabla p = \underset{\sim}{B}$.

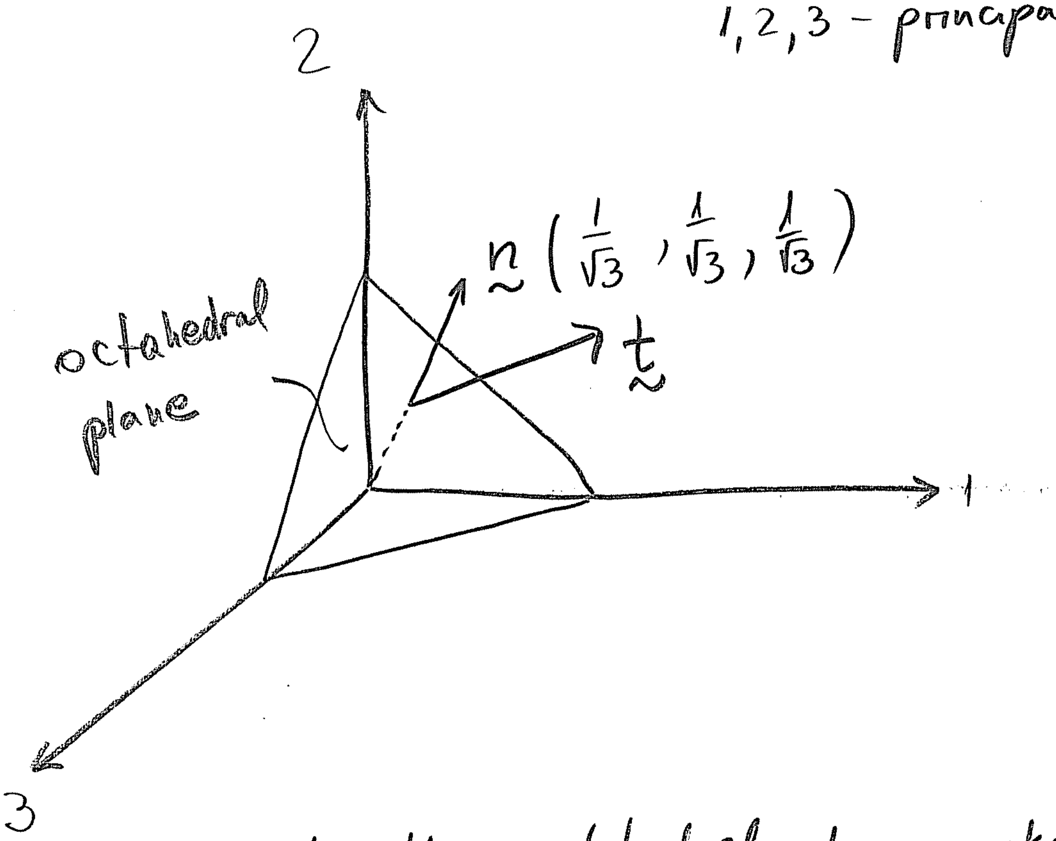
$\underset{\sim}{B}$ = body force

4) Prove that in the case of plane stress case with body forces (V is the body force potential), the biharmonic equation that must be satisfied by the Airy stress function Φ is

$$\nabla^4\Phi + (1-\nu)\nabla^2 V = 0$$

① Octahedral normal and shear stresses

1, 2, 3 - principal directions



The normal $\underset{\sim}{n}$ to the octahedral plane makes equal angles with principal directions 1, 2, 3

$\underset{\sim}{\sigma}$ = stress tensor

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_z & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The traction $\underset{\sim}{t}$ on the octahedral plane is

$$\underset{\sim}{t} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{\sigma_1}{\sqrt{3}} \\ \frac{\sigma_2}{\sqrt{3}} \\ \frac{\sigma_3}{\sqrt{3}} \end{bmatrix}$$

The normal stress on the octahedral plane is:

$$t_n = \underset{\sim}{t}_n^T \cdot \underset{\sim}{n} = \begin{bmatrix} \frac{\sigma_1}{\sqrt{3}} & \frac{\sigma_2}{\sqrt{3}} & \frac{\sigma_3}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

and it acts in the direction of $\underset{\sim}{n}$

The shear stress on the octahedral plane:

$$t_s^2 = \underset{\sim}{t} \cdot \underset{\sim}{t} - t_n^2 = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3} - \left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2 =$$

$$= \frac{2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_3 - 2\sigma_1\sigma_3}{9}$$

$$= \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{9}$$

$$= \frac{2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_1\sigma_3 + 2\sigma_2\sigma_3)}{9} -$$

$$- \frac{6(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)}{9} = \frac{2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)}{9}$$

Notice that t_n and t_s are in terms of principal stresses, and we want them in terms of stress components in the xyz system of reference. For this we use stress invariants:

The normal stress on the octahedral plane is:

$$t_n = \underset{\sim}{t}^T \cdot \underset{\sim}{n} = \begin{bmatrix} \frac{\sigma_1}{\sqrt{3}} & \frac{\sigma_2}{\sqrt{3}} & \frac{\sigma_3}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{\sigma_1}{3} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3}$$

and it acts in the direction of $\underset{\sim}{n}$

The shear stress on the octahedral plane is:

$$t_s = \sqrt{\underset{\sim}{t} \cdot \underset{\sim}{t} - t_n^2} = \sqrt{\frac{\sigma_1^2}{3} + \frac{\sigma_2^2}{3} + \frac{\sigma_3^2}{3} - \left(\frac{\sigma_1}{3} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3}\right)^2}$$

$$= \sqrt{\frac{2\sigma_1^2}{9} + \frac{2\sigma_2^2}{9} + \frac{2\sigma_3^2}{9} - \frac{2}{9}\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)}$$

Notice that t_n and t_s are in terms of principal stresses, and we want them in terms of the stress components in the xyz reference system. For this we use stress invariants:

$$t_n = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$t_s = \sqrt{\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3} - \left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2}$$

Notice that

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = I_2$$

But I_2 can also be expressed in the xyz system:

$$t_n = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{\sigma_x+\sigma_y+\sigma_z}{3} = I_1$$

$$t_s^2 = \frac{2(\sigma_1+\sigma_2+\sigma_3)^2 - 6(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)}{9} =$$

$$= \frac{2I_1^2 - 6I_2}{9}$$

Recall,

$$I_1 = \sigma_x+\sigma_y+\sigma_z$$

$$I_2 = \sigma_x\sigma_y+\sigma_x\sigma_z+\sigma_y\sigma_z-\tau_{xy}^2-\tau_{xz}^2-\tau_{yz}^2$$

Then,

$$t_s^2 = \frac{1}{9}\left[2(\sigma_x+\sigma_y+\sigma_z)^2 - 6(\sigma_x\sigma_y+\sigma_x\sigma_z+\sigma_y\sigma_z-\tau_{xy}^2-\tau_{yz}^2-\tau_{xz}^2)\right]$$

$$= \frac{1}{9}\left[2(\sigma_x^2+\sigma_y^2+\sigma_z^2) - 2(\sigma_x\sigma_y+\sigma_x\sigma_z+\sigma_y\sigma_z) + 6(\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2)\right]$$

$$= \frac{1}{9}\left[(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2)\right]$$

Thus

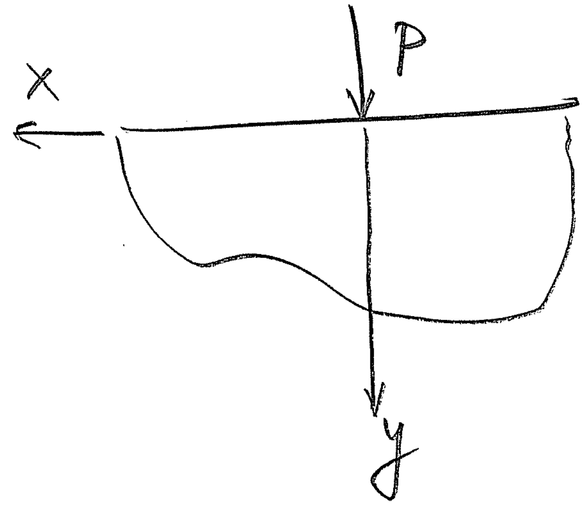
$$t_s = \frac{1}{3}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2+6\tau_{xy}^2+6\tau_{yz}^2+\tau_{xz}^2}$$

(2)

$$\sigma_x = -\frac{2Px^2y}{\pi(x^2+y^2)^2}$$

$$\sigma_y = -\frac{2Py^3}{\pi(x^2+y^2)^2}$$

$$\tau_{xy} = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}$$



$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\frac{\sigma_x+\sigma_y}{2} = -\frac{2Py(x^2+y^2)}{2\pi(x^2+y^2)^2} = -\frac{Py}{\pi(x^2+y^2)}$$

$$\frac{\sigma_x-\sigma_y}{2} = \frac{1}{2}\left(\frac{2Py^3}{\pi(x^2+y^2)^2} - \frac{2Px^2y}{\pi(x^2+y^2)^2}\right] = \frac{Py(y^2-x^2)}{\pi(x^2+y^2)^2}$$

$$\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2 = \left[\frac{Py(y^2-x^2)}{\pi(x^2+y^2)^2}\right]^2 + \left[\frac{2Pxy^2}{\pi(x^2+y^2)^2}\right]^2 =$$

$$= \left[\frac{Py}{\pi(x^2+y^2)^2}\right]^2\left[(y^2-x^2)^2 + 4x^2y^2\right] =$$

$$= \left[\frac{Py}{\pi(x^2+y^2)^2}\right]^2 (x^2+y^2)^2$$

$$\sigma_{1,2} = -\frac{Py}{\pi(x^2+y^2)} \pm \frac{Py}{\pi(x^2+y^2)}$$

Rearranging for $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

$$\sigma_3 = -\frac{2Py}{\pi(x^2+y^2)} \qquad \sigma_2 = 0 \qquad \sigma_1 = 0$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{Py}{\pi(x^2+y^2)}$$

③

$$\underset{\sim}{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

$p = p(x, y, z)$

Equilibrium :

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + B_x = 0 & (1) \\[2ex] \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + B_y = 0 & (2) \\[2ex] \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + B_z = 0 & (3) \end{cases}$$

$(1) \iff -\dfrac{\partial p}{\partial x} + B_x = 0 \quad \Big| \cdot \underset{\sim}{\hat{i}}$

$(2) \iff -\dfrac{\partial p}{\partial y} + B_y = 0 \quad \Big| \cdot \underset{\sim}{j}$ $\Rightarrow$ by adding (1)+(2)+(3)

$(3) \iff -\dfrac{\partial p}{\partial z} + B_z = 0 \quad \Big| \cdot \underset{\sim}{k}$

$$\Rightarrow -\left(\frac{\partial p}{\partial x}\underset{\sim}{i} + \frac{\partial p}{\partial y}\underset{\sim}{j} + \frac{\partial p}{\partial z}\underset{\sim}{k}\right) + \left(B_x\underset{\sim}{i} + B_y\underset{\sim}{j} + B_z\underset{\sim}{k}\right) = 0$$

$$-\underset{\sim}{\nabla} p + \underset{\sim}{B} = 0 \quad \Rightarrow \quad \boxed{\underset{\sim}{\nabla} p = \underset{\sim}{B}}$$

(4) 2D stress state

Compatibility of strains:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \qquad (1)$$

Hooke's law:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right] \qquad (2)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right] \qquad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)\tau_{xy}}{E} \qquad (4)$$

Introduce (2), (3) and (4) into (1):

$$\frac{1}{E}\left[\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right] - 2\frac{(1+\nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \qquad (5)$$

Stress equilibrium:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0 \qquad (6)$$

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \sigma_y}{\partial y} + B_y = 0 \qquad (7)$$

Let $B_x = -\dfrac{\partial V}{\partial x}$

$B_y = -\dfrac{\partial V}{\partial y}$

$$(6) \Rightarrow \frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad (8)$$

$$(7) \Rightarrow \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial}{\partial y}(\sigma_y - V) = 0 \qquad (9)$$

$$(8) \Rightarrow \frac{\partial}{\partial x}(\sigma_x - V) = -\frac{\partial \tau_{xy}}{\partial y} \Rightarrow$$

$\Rightarrow \exists\ \phi'$ - scalar function such that

$$\sigma_x - V = \frac{\partial \phi'}{\partial y}$$

$$-\tau_{xy} = \frac{\partial \phi'}{\partial x} \qquad (*)$$

$$(9) \Rightarrow \frac{\partial}{\partial y}(\sigma_y - V) = -\frac{\partial \tau_{xy}}{\partial x}$$

$\Rightarrow \exists\ \phi''$ - scalar function such that

$$\sigma_y - V = \frac{\partial \phi''}{\partial x}$$

$$-\tau_{xy} = \frac{\partial \phi''}{\partial y} \qquad (**)$$

From $(*)$ and $(**)$ $\Rightarrow$

$\Rightarrow \dfrac{\partial \phi'}{\partial x} = \dfrac{\partial \phi''}{\partial y} \Rightarrow \exists\ \phi$ - scalar function,

such that

$$\phi' = \frac{\partial \phi}{\partial y} \qquad \phi'' = \frac{\partial \phi}{\partial x}$$

Then

$$\sigma_x - V = \frac{\partial \phi'}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} \Rightarrow \sigma_x = \frac{\partial^2 \phi}{\partial y^2} + V$$

$$\sigma_y - V = \frac{\partial \phi''}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} \Rightarrow \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V$$

$$\sigma_{xy} = -\frac{\partial \phi'}{\partial x} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Thus, we obtained

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + V \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y}$$

Introduce these into eq. (5) after simplifying by E:

$$\frac{\partial^4\phi}{\partial y^4} + \frac{\partial^2 V}{\partial y^2} - \nu\left(\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^2 V}{\partial y^2}\right) + \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^2 V}{\partial x^2} -$$

$$-\nu\left(\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^2 V}{\partial x^2}\right) + 2(1+\nu)\frac{\partial^4\phi}{\partial x^2\partial y^2} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial x^4} - 2\nu\frac{\partial^4\phi}{\partial x^2\partial y^2} + 2(1+\nu)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} -$$

$$-\nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} + (1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \boxed{\nabla^4\phi + (1-\nu)\nabla^2 V = 0}$$