ME 548 – Elasticity

Homework 2

Due: Friday, Sept. 27, 2013

1) Problem 2.4.1 p. 75.
2) Problem 2.4.3 p. 75.
3) Problem 2.4.6 p. 76.
4) Problem 2.6.1 p. 85.
5) Problem 2.6.3 p. 86.
6) Problem 2.8.2 p. 94.
7) Problem 2.8.3 p. 94.

2-4.1

$$J = \begin{vmatrix} 1-k & k & 0 \\ k & 1-k & 0 \\ kx_3 & 0 & 1+kx_1 \end{vmatrix} = (1-2k)(1+kx_1) > 0$$

For admissible deformation, we take

$1 > 2k \quad \therefore \quad k < \frac{1}{2} \quad$ and $\quad kx_1 > -1$

2-4.3

$$J = \begin{vmatrix} -\frac{(c-y)}{c}\sin\frac{x}{c} & -\cos\frac{x}{c} & 0 \\ \frac{c-y}{c}\cos\frac{x}{c} & -\sin\frac{x}{c} & 0 \\ 0 & 0 & 1 \end{vmatrix} > 0$$

$c \neq 0$

$\therefore\ 1 - \frac{y}{c} > 0$. For $c > 0$, $c > y$.

$\therefore\ c > h$. For $c < 0$, $c < -h$.

2-4.6

$$
\begin{aligned}
I_2\left(\bar{C}\right) &= \bar{C}_{ij}\bar{C}_{ji} \\
&= a_{i\alpha}a_{j\beta}a_{j\gamma}a_{i\kappa}C_{\alpha\beta}C_{\gamma\kappa} \\
&= a_{\kappa\alpha}a_{\gamma\beta}C_{\alpha\beta}C_{\gamma\kappa} \\
&= C_{\alpha\beta}C_{\alpha\beta} = I_2\left(\boldsymbol{C}\right)
\end{aligned}
$$

$$
\begin{aligned}
I_3\left(\bar{C}\right) &= \bar{C}_{ij}\bar{C}_{jk}\bar{C}_{ki} \\
&= a_{i\alpha}a_{j\beta}a_{j\gamma}a_{k\kappa}a_{k\chi}a_{i\eta}C_{\alpha\beta}C_{\gamma\kappa}C_{\chi\eta} \\
&= a_{\alpha\eta}a_{\gamma\beta}a_{\chi\kappa}C_{\alpha\beta}C_{\gamma\kappa}C_{\chi\eta} \\
&= C_{\alpha\beta}C_{\beta\chi}C_{\chi\alpha} = I_3\left(\boldsymbol{C}\right)
\end{aligned}
$$

2-6.1

By Eq. 2-6.14, given data yields

$$0.004 = 0.002\left(\tfrac{4}{5}\right) + 0.002(0) + (-.002)\left(\tfrac{1}{5}\right) + + 2\epsilon_{12}(0) + 2\epsilon_{13}\left(\tfrac{2}{5}\right) + 2\epsilon_{23}(0)$$

$$\therefore \boxed{\epsilon_{13} = 0.0035}$$

$$0.003 = 0.002\left(\tfrac{9}{10}\right) + 0.002\left(\tfrac{1}{10}\right) + 2\epsilon_{12}\left(-\tfrac{3}{10}\right)$$

$$\therefore \boxed{\epsilon_{12} = -0.00166\overline{6}}$$

$$0.001 = .002\left(\tfrac{1}{3}\right) + .002\left(\tfrac{1}{3}\right) + (-.002)\left(\tfrac{1}{3}\right) + + 2\epsilon_{12}\left(\tfrac{1}{3}\right) + 2\epsilon_{13}\left(\tfrac{1}{3}\right) + 2\epsilon_{23}\left(\tfrac{1}{3}\right)$$

$$\therefore \boxed{\epsilon_{23} = -0.00133\overline{3}}$$

2-6.3

By Eq. 2-6.7, and given data

$$d\xi_1 = (1+6kx_1)dx_1 + k\,dx_2$$

$$d\xi_2 = (1+4kx_2)dx_2 + k\,dx_3$$

$$d\xi_3 = k\,dx_1 + (1+8kx_3)dx_3$$

$$(ds)^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2$$

$$= [(1+6kx_1)dx_1 + k\,dx_2]^2 + [(1+4kx_2)dx_2 + k\,dx_3]^2 + [k\,dx_1 + (1+8kx_3)dx_3]^2$$

$$MF = \frac{1}{2}\left[\left(\frac{ds}{dS}\right)^2 - 1\right] = \frac{1}{2}\Big\{[(1+6kx_1)n_1 + kn_2]^2 + [(1+4kx_2)n_2 + kn_3]^2 + [(1+8kx_3)n_3 + kn_1]^2 - 1\Big\}$$

Or

$$MF = \frac{1}{6}[42k + 155k^2] = 7k + \frac{155}{6}k^2$$

2-8.2

a) $J = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3(2+1) = 9 > 0$

b) strain $\equiv MF = \epsilon_{\alpha\beta} N_\alpha N_\beta$

$(N_1, N_2, N_3) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$\epsilon_{11} = 4,\ \epsilon_{22} = 1,\ \epsilon_{33} = 2,\ \epsilon_{12} = \frac{3}{2},\ \epsilon_{23} = -\frac{1}{2};\ \epsilon_{13} = 0$

$\therefore MF = (4+1+2+3-1)\frac{1}{3} = 3$

$(\frac{ds}{ds})^2 = 1 + 2MF = 7 \quad ; \quad ds \approx 2.65\, ds$

Relative elongation $e_i = \frac{ds - ds}{ds} \cong 1.65$

c) $\eta_1 = \eta_2 = 0,\ \eta_3 = 1$ Hence, by Eq. (2-8.1)

$3N_1 + N_2 = 0 \implies N_1 = -\frac{N_2}{3}$

$N_2 + N_3 = 0 \implies N_3 = -N_2$

$-N_2 + 2N_3 = \sqrt{1+2MF}$ MF Not Known

From $N_1^2 + N_2^2 + N_3 = 1$; $N_2^2 = \frac{9}{19}$. Hence,

$N_1^2 = \frac{1}{19}$; $N_3^2 = \frac{9}{19}$. Then, $\sqrt{1+2MF} = -N_2 + 2N_3$

$= \mp\frac{3}{\sqrt{19}} \mp \frac{6}{\sqrt{19}} = \mp\frac{9}{\sqrt{19}}$. $1 + 2MF = \frac{81}{19}$

Or $MF \cong 1.63$

d) Initial angle θ given by $\cos\theta = m_\alpha n_\alpha$

$= (1)(\frac{1}{\sqrt{3}}) + (0)(\frac{1}{\sqrt{3}}) + (0)(\frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}}$ $\theta = 54.74°$

By Eq. 2-8.3, final angle ϑ given

by $\sqrt{(1+2MF_1)(1+2MF_2)}\ \cos\vartheta = \cos\theta + 2\epsilon_{\alpha\beta} m_\alpha n_\beta$

By Eq 2-6.13, $MF_1 = \epsilon_{11} n_1^2 = 4$ and $MF_2 = 3$

from part b. Therefore;

$\sqrt{63}\cos\vartheta = \frac{1}{\sqrt{3}} + 2[4 + \frac{3}{2}]\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} + \frac{11}{\sqrt{3}} = 4\sqrt{3}$

$\cos\vartheta = \frac{4}{\sqrt{21}} = 0.871$; $\vartheta = 29.21°$

$\therefore \vartheta - \theta = 29.21° - 54.74° = -25.53°$

2-8.3

a) $J = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3(8+1) = 27 > 0$

b) $\epsilon_{11} = 4$, $\epsilon_{22} = 8$, $\epsilon_{33} = 2$, $\epsilon_{23} = 1$, $\epsilon_{12} = \epsilon_{13} = 0$

$MF = \epsilon_{\alpha\beta} N_\alpha N_\beta = (4+8+2+2)\frac{1}{3} = \frac{16}{3}$

$\Gamma = \cos\theta + 2\epsilon_{\alpha\beta} M_\alpha N_\beta$

$\cos\theta = \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}(0) = 0$

$\theta = \frac{\pi}{2} \quad \therefore \Gamma = 2\epsilon_{\alpha\beta} M_\alpha N_\beta$

$\Gamma = 2\left[(4)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) + (8)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{3}}\right)(0) + (1)\left(\frac{1}{\sqrt{3}}\right)(0) + (1)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right)\right]$

$= \frac{10}{\sqrt{6}} = 4.0825$

c) $\eta_1 = \eta_3 = 0 \quad \eta_2 = 1$

$\eta_\alpha = \sqrt{1+2MF} = (\delta_{\alpha\beta} + u_{\alpha,\beta})N_\beta \qquad 0 = 3N_1 \quad \therefore N_1 = 0$

$\sqrt{1+2MF} = 4N_2 + N_3$

$0 = -N_2 + 2N_3 \implies N_2 = 2N_3$

$\therefore N_1^2 + N_2^2 + N_3^2 = 5N_3^2 = 1 \quad ; \quad N_3^2 = \frac{1}{5}, \quad N_2^2 = \frac{4}{5}$

$\therefore 1 + 2MF = \frac{81}{5}, \quad \underline{MF = 7.6}$