ME 548 – Elasticity

Homework 5

Due: Fri., Nov. 8, 2013

1) Problem 4-6-2 p. 262.
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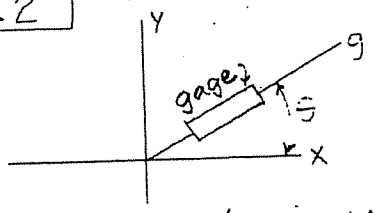
3) Problem 4-6-10 p. 264.
4) Problem 4-7-2 p. 268.

4-6.2



$\ell = \cos\theta, \quad m = \sin\theta, \quad n = 0$

or $N_1 = \cos\theta, \; N_2 = \sin\theta, \; N_3 = 0$

By Eq (2-6.14), $MF_g = \epsilon_g + \frac{1}{2}\epsilon_g^2 (\cong \epsilon_g) = \epsilon_{\alpha\beta} N_\alpha N_\beta$

Hence, $\epsilon_g = \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + 2\epsilon_{12}\sin\theta\cos\theta$

$= \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$

By stress-strain relations,

$$\sigma_x = (\lambda + 2G)\epsilon_x + \lambda(\epsilon_y + \epsilon_z)$$

$$\sigma_y = (\lambda + 2G)\epsilon_y + \lambda(\epsilon_x + \epsilon_z)$$

$$\sigma_z = (\lambda + 2G)\epsilon_z + \lambda(\epsilon_y + \epsilon_x) = 0, \text{ since surface unloaded}$$

$\tau_{xy} = G\gamma_{xy} = 0$, since (x, y) are principal axes

$\tau_{xz} = \tau_{yz} = 0$ since surface unloaded

By $\sigma_z = 0$, $\epsilon_z = -\frac{\lambda}{\lambda + 2G}(\epsilon_x + \epsilon_y)$ and then

$$\sigma_x = \frac{4G(\lambda + G)}{\lambda + 2G}\epsilon_x + \frac{2\lambda G}{\lambda + 2G}\epsilon_y$$

Also since $\gamma_{xy} = 0$, $\epsilon_g = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta$

Then since we want $\sigma_x = K\epsilon_g$, we have

$$K\cos^2\theta = \frac{4G(\lambda + G)}{\lambda + 2G} = \frac{E}{1 - \nu^2}$$

$$K\sin^2\theta = \frac{2G\lambda}{\lambda + 2G} = \frac{\nu E}{1 - \nu^2}$$

or $\tan^2\theta = \nu$ or $\tan\theta = \sqrt{\nu}$, $K = \frac{E}{1 - \nu}$

4-6.6

By Fig P4-6.6

$l_1, m_1, n_1 = \cos\theta, \sin\theta, 0$

$l_2, m_2, n_2 = -\cos\theta, \sin\theta, 0$

$l_3, m_3, n_3 = 1, 0, 0$

$\therefore \quad \epsilon_3 = \epsilon_x l_3^2 + \cdots = \epsilon_x$

$\epsilon_1 = \epsilon_x l_1^2 + \epsilon_y m_1^2 + \epsilon_z n_1^2 + l_1 m_1 \gamma_{xy} + \cdots$

$\epsilon_2 = \epsilon_x l_2^2 + \epsilon_y m_2^2 \quad \epsilon_z n_2^2 + l_2 m_2 \gamma_{xy} + \cdots$

or $\epsilon_1 = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta$

$\epsilon_2 = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta - \gamma_{xy} \sin\theta \cos\theta$

Hence,

$\epsilon_1 + \epsilon_2 - 2\epsilon_3 \cos^2\theta = 2\epsilon_y \sin^2\theta$

or $\epsilon_y = \dfrac{\epsilon_1 + \epsilon_2 - 2\epsilon_3 \cos^2\theta}{1 - \cos 2\theta}$

also $\epsilon_1 - \epsilon_2 = \gamma_{xy} 2\sin\theta\cos\theta = \gamma_{xy} \sin 2\theta$

$\therefore \quad \gamma_{xy} = \dfrac{\epsilon_1 - \epsilon_2}{\sin 2\theta}$

Since the surface is unloaded, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ and by stress-strain relations $\gamma_{xz} = \gamma_{yz} = 0$, $\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$

$= -\frac{\nu}{E}(\sigma_x + \sigma_y)$.

Also, $\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$, $\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$

$\epsilon_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)$

$\therefore \quad \sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) = \dfrac{E\left[\epsilon_3(1-\cos 2\theta) + \nu(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_3\cos 2\theta)\right]}{(1-\nu^2)(1-\cos 2\theta)}$

$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) = \dfrac{E\left[\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_3\cos 2\theta + \nu\epsilon_3(1-\cos 2\theta)\right]}{(1-\nu^2)(1-\cos 2\theta)}$

$\tau_{xy} = G\gamma_{xy} = \dfrac{G(\epsilon_1 - \epsilon_2)}{\sin 2\theta}$

$\epsilon_z = -\dfrac{\nu(\epsilon_1 + \epsilon_2 - 2\epsilon_3\cos 2\theta)}{(1-\nu)(1-\cos 2\theta)}$

4-6.10

For a: $l_a = 1$, $m_a = n_a = 0$

For b: $l_b = \frac{1}{\sqrt{2}}$, $m_b = \frac{1}{\sqrt{2}}$, $n_b = 0$

For c: $l_c = 0$, $m_c = 1$, $n_c = 0$

$\therefore \epsilon_a = 0.0002 = \epsilon_x l_a^2 + \ldots = \epsilon_x \quad \therefore \underline{\epsilon_x = 0.0002}$

$\epsilon_b = 0.0001 = \epsilon_x l_b^2 + \epsilon_y m_b^2 + \ldots = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}\gamma_{xy}$

$\epsilon_c = 0.0004 = \epsilon_x l_c^2 + \epsilon_y m_c^2 + \ldots = \epsilon_y \quad \therefore \epsilon_y = 0.0004$

Hence, $\gamma_{xy} = 0.0002 - \epsilon_x - \epsilon_y = -0.0004$

To determine principal stresses and their directions relative to a, we can determine principal strains for the plane and use stress strain relations (noting that since $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, then $\gamma_{xz} = \gamma_{yz} = 0$ and $\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{\nu}{E}(\sigma_1 + \sigma_2)$ is a principal strain where σ_1, σ_2 are principal stresses in the a, b plane). Since $\sigma_z = 0$,

$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) = 3200$, $\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) = 4800$

$\sigma_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{0.25}{10^7} \times 8000 = -0.0002$ with direction cosines $l_z = 0$, $m_z = 0$, $n_z = \pm 1$

4-6.10 cont'd

The strain invariants are $J_1 = \epsilon_x + \epsilon_y + \epsilon_z = 0.0004$

$J_2 = \epsilon_x\epsilon_y + \epsilon_x\epsilon_z + \epsilon_y\epsilon_z - \frac{1}{4}\gamma_{xy}^2 = -8\times10^{-8}$

$$J_3 = \begin{vmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{vmatrix} = -8\times10^{-12}$$

$\therefore$ The principal strains are the roots of

$\epsilon^3 - 4\times10^{-4}\epsilon^2 - 8\times10^{-8}\epsilon + 8\times10^{-12} = 0$ Let unit of strain be 0.0001. Then $(\epsilon+2)(\epsilon-3-\sqrt{5})(\epsilon-3+\sqrt{5}) = 0$

and $\epsilon_1 = 3+\sqrt{5}$, $\epsilon_2 = 3-\sqrt{5}$, $\epsilon_3 = -2$ or

$[\epsilon_1 = 5.236\times10^{-4},\ \epsilon_2 = 0.764\times10^{-4},\ \epsilon_3 = -2.000\times10^{-4}]$

$\therefore$ $\sigma_1 = \frac{E}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2) = 5789$ psi

$\sigma_2 = \frac{E}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1) = 2211$ psi

For $\epsilon_1 = 3+\sqrt{5}$, the direction cosines are given by

$(2-3-\sqrt{5})\,l_1 - 2m_1 = 0 \Rightarrow m_1 = -\frac{1+\sqrt{5}}{2}\,l_1$

$-2l_1 + (4-3-\sqrt{5})\,m_1 = 0 \Rightarrow$ checks O.K.

$l_1^2 + m_1^2 = 1 \Rightarrow l_1 = \pm\sqrt{\frac{2}{5+\sqrt{5}}}$

$\therefore$ $l_1 = \pm\sqrt{\frac{2}{5+\sqrt{5}}} = \pm 0.5257$, $m_1 = \mp\sqrt{\frac{3+\sqrt{5}}{5+\sqrt{5}}} = \mp 0.8506$

For $\epsilon_2 = 3-\sqrt{5}$,

$(2-3+\sqrt{5})\,l_2 - 2m_2 = 0 \Rightarrow m_2 = -\frac{1-\sqrt{5}}{2}\,l_2$

$-2l_2 + (4-3+\sqrt{5})\,m_2 = 0 \Rightarrow$ checks O.K.

$l_2^2 + m_2^2 = 1 \Rightarrow l_2 = \pm\sqrt{\frac{2}{5-\sqrt{5}}}$

$\therefore$ $l_2 = \pm\sqrt{\frac{2}{5-\sqrt{5}}} = \pm 0.8506$, $m_2 = \mp\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} = \mp 0.5257$

cont'd

4-7.2

(a) <u>caution</u>: To avoid arriving at the erroneous result that $\sigma_{33}=0$ for plane strain, it is necessary to apply the relation $\sigma_\alpha = \frac{\partial U}{\partial e_\alpha}$ before setting $\epsilon_{33}=0$. Thus with $\sigma_1=\sigma_{11}$, $\sigma_2=\sigma_{22}$, etc. (see Eq. 9 and Eq 4-3.1) we obtain

$$\sigma_{11} = \lambda(\epsilon_{11}+\epsilon_{22}) + 2G\epsilon_{11}$$

$$\sigma_{22} = \lambda(\epsilon_{11}+\epsilon_{22}) + 2G\epsilon_{22}$$

$$\sigma_{33} = \lambda(\epsilon_{11}+\epsilon_{22}),\quad \sigma_{12}=2G\epsilon_{12},\quad \sigma_{13}=\sigma_{23}=0$$

(b) as in part (a) we first employ the formula $\sigma_\alpha = \frac{\partial U}{\partial e_\alpha}$ and then set $\sigma_{33}=\sigma_{13}=\sigma_{23}=0$

Thus we obtain

$$\sigma_{11} = \lambda(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) + 2G\epsilon_{11}$$

$$\sigma_{22} = \lambda(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) + 2G\epsilon_{22}$$

$$\sigma_{33} = \lambda(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) + 2G\epsilon_{33} = 0$$

$$\therefore\ \epsilon_{33} = -\frac{\lambda}{\lambda+2G}(\epsilon_{11}+\epsilon_{22}) = -\frac{\nu}{1-\nu}(\epsilon_{11}+\epsilon_{22})$$

$$\sigma_{12}=2G\epsilon_{12},\quad \sigma_{13}=0\ (\therefore \epsilon_{13}=0),\quad \sigma_{23}=0\ (\therefore \epsilon_{23}=0)$$

Alternatively, after elimination of ϵ_{33}, we may write

$$\sigma_{11} = \frac{4G(\lambda+G)}{\lambda+2G}\epsilon_{11} + \frac{2G\lambda}{\lambda+2G}\epsilon_{22} = \frac{E}{1-\nu^2}(\epsilon_{11}+\nu\epsilon_{22})$$

$$\sigma_{22} = \frac{2G\lambda}{\lambda+2G}\epsilon_{11} + \frac{4G(\lambda+G)}{\lambda+2G}\epsilon_{22} = \frac{E}{1-\nu^2}(\nu\epsilon_{11}+\epsilon_{22})$$

$$\sigma_{12} = 2G\epsilon_{12} = \frac{E}{1+\nu}\epsilon_{12}$$