ME 548 – Elasticity

Homework 6 (optional)

Due: Thursday, Dec. 12, 2013

1) Problem 5-4-12 (a) and (b) p. 396.
2) Problem 5-4-15 p. 397.

5-4.12

(a) With $F = C_1 x y^3 + C_2 x y$, we find

$\sigma_x = F_{yy} = 6C_1 xy$, $\tau_{xy} = -F_{xy} = -3C_1 y^2 - C_2$

$\sigma_y = F_{xx} = 0$

Since the cantilever is loaded by a shear force at one end, on this end σ_x must be zero. Therefore let the cantilever beam be supported at the section $x = L$ and lie in the region $0 \le x \le L$, $-c \le y \le c$, with the shear force applied at the section $x = 0$ and directed in the negative y sense. (See Fig. P4-18.6) Also for simplicity let the beam width be 1 unit. Then

$$P = \int_{-c}^{c} \tau_{xy}(1)\,dy = \int_{-c}^{c} (-3C_1 y^2 - C_2)\,dy = -C_1 y^3 - C_2 y \Big|_{-c}^{c}$$

or $P = -2C_1 c^3 - 2C_2 c$ (a)

For moment equilibrium, we have

$$\Sigma M\Big|_{x=L} = PL - \int_{-c}^{c} \sigma_x y(1)\,dy = 0$$

or $PL = \int_{-c}^{c} 6C_1 x y^2\,dy \Big|_{x=L} = 2C_1 L y^3 \Big|_{-c}^{c} = 4C_1 L c^3$ (b)

$\therefore\ C_1 = \dfrac{P}{4c^3}$, $C_2 = \dfrac{3P}{4c}$ (c)

$\sigma_x = \dfrac{3}{2c^3} Pxy$, $\tau_{xy} = \dfrac{3P}{4c^3}(c^2 - y^2)$ (d)

(b) By the strain-displacement and strain-stress relations, we have

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{3}{2}\frac{Pxy}{Ec^3} \quad \therefore \quad u = \frac{3}{4}\frac{Px^2y}{Ec^3} + f(y)$$

$$\epsilon_y = \frac{\partial v}{\partial y} = -\frac{3}{2}\frac{\nu Pxy}{Ec^3} \quad \therefore \quad v = g(x) - \frac{3}{4}\frac{\nu Pxy^2}{Ec^3}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{3}{4}\frac{P}{Gc^3}(c^2 - y^2) \quad ; \quad G = \frac{E}{2(1+\nu)}$$

$$\therefore \frac{3}{4}\frac{Px^2}{Ec^3} + f'(y) + g'(x) - \frac{3}{4}\frac{\nu P}{Ec^3}y^2 = \frac{3}{4}\frac{P}{c^3\left[\frac{E}{2(1+\nu)}\right]}(c^2 - y^2) = \frac{3}{2}\frac{P(1+\nu)}{Ec^3}(c^2 - y^2)$$

$$\therefore g'(x) + \frac{3}{4}\frac{Px^2}{Ec^3} = -f'(y) + \frac{3}{2}\frac{P(1+\nu)}{Ec^3}(c^2 - y^2) + \frac{3}{4}\frac{\nu Py^2}{Ec^3} = C_1$$

Hence, $g(x) = -\frac{1}{4}\frac{Px^3}{Ec^3} + C_1x + C_2$

$$f(y) = \frac{3}{2}\frac{P(1+\nu)}{Ec^3}\left(c^2y - \frac{y^3}{3}\right) - C_1y + C_3 + \frac{1}{4}\frac{\nu Py^3}{Ec^3}$$

$$\therefore \quad u = \frac{3}{4}\frac{Px^2y}{Ec^3} + \frac{3}{2}\frac{P(1+\nu)}{Ec^3}\left(c^2y - \frac{y^3}{3}\right) - C_1y + C_3 + \frac{1}{4}\frac{\nu Py^3}{Ec^3}$$

$$v = -\frac{1}{4}\frac{Px^3}{Ec^3} + C_1x + C_2 - \frac{3}{4}\frac{\nu Pxy^2}{Ec^3} \qquad (e)$$

(5.4.-15) $F = Ax^3y$

For place elastic problems, $\sigma_x = \frac{\partial^2 F}{\partial y^2} = 0$, $\sigma_y = \frac{\partial^2 F}{\partial x^2} = 6Axy$

$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -3Ax^2$ (a)

∴ For $x \doteq -a$ or $+a$, the boundary conditions are $\sigma_x = 0$, $\tau_{xy} = -3Aa^2$

For $y = -b$, $\sigma_y = -6Abx$, $\tau_{xy} - 3Ax^2$ (b)

For $y = +b$, $\sigma_y = 6Abx$, $\tau_{xy} = -3Ax^2$

Therefore the region $-a \leq x \leq a$, $-b \leq y \leq b$ is subjected to boundary stresses given by Eq. (b) and the stresses are determined by Eq.(a)