

That which surpasses all understanding **Mathematical insights on the limitations of human thought**

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One generation passeth away, and another generation cometh: but the earth abideth for ever. The sun also ariseth, and the sun goeth down, and hasteth to his place where he arose. The wind goeth toward the south, and turneth about unto the north; it whirleth about continually, and the wind returneth again according to his circuits. All the rivers run into the sea; yet the sea is not full; unto the place from whence the rivers come, thither they return again. All things are full of labour; man cannot utter it: the eye is not satisfied with seeing, nor the ear filled with hearing.

Ecclesiastes 1:4-8

I remember those verses striking a powerful chord within me when I read them on a bright autumn day in 1980. I was then in the first few months of my church mission in central Virginia. But reading those words took my mind and emotions back to the desert mountains of western Utah earlier that year. A friend and I had taken a quick camping trip to collect fossils in that remote area, and something in the desert sun, the bare exposure of earth, and the surrounding evidence of unimaginably ancient life produced a feeling so strong that I recognized it immediately when I later stumbled on that passage of scripture. I couldn't then put my finger on the exact meaning of the emotion – something about the smallness of our place in the universe and our inability to understand it all. It was as powerful as any religious feeling I had ever had, and its duplication at reading the opening of Ecclesiastes nearly brought me to tears. I read the remainder of the book eagerly, naively hoping to find its resolution.

There are other more well-known scriptures that express related sentiments: Psalm 8 has the famous poetic lines

*When I consider thy heavens, the work of thy fingers,
the moon and the stars, which thou hast ordained;
What is man, that thou art mindful of him?*

And in the Pearl of Great Price we read that

. . . Moses looked, and beheld the world upon which he was created; and Moses beheld the world and the ends thereof, and all the children of men which are, and which were created; of the same he greatly marveled and wondered. . . and he said unto himself: Now, for this cause I know that man is nothing, which thing I never had supposed.
(Moses 1:8,10)

As a missionary I was aware that these passages are well-used because they *do* resolve. Both respond to the image of our apparent insignificance with verses affirming our central role in God's plan, but neither dwells much on the inadequacy of understanding that inspired the feeling in the first place. The Moses story does contain some interesting references to it, such as God proclaiming of his creations that “. . . *they cannot be numbered unto man; but they are numbered unto me, for they are mine*” (verse 37) and “*Here is wisdom and it remaineth in me*” (verse 31).

Similar wording is found in Joseph Smith's reaction to his own vision of God's kingdoms: “*the mysteries of his kingdom ... surpass all understanding...*” (D&C 76:114). All these hint at the theme of another beautiful Old Testament passage I have loved since the first time I read it:

For my thoughts are not your thoughts, neither are your ways my ways, saith the LORD. For as the heavens are higher than the earth, so are my ways higher than your ways, and my thoughts than your thoughts. (Isaiah 55:8-9)

The book of Ecclesiastes, which I have loved ever since that first careful reading on my mission, grapples with the issue for all of its chapters, coming only to the less-than-satisfying conclusion that we *cannot* ever understand:

All this have I proved by wisdom: I said, I will be wise; but it was far from me. That which is far off, and exceeding deep, who can find it out? (Ecclesiastes 7:23-24)

This sentiment runs counter in spirit (though not contradictory in implication) to some well-known Mormon themes: “*The glory of God is intelligence*” (D&C 93:36) and the strikingly specific passage

Whatever principle of intelligence we attain unto in this life, it will rise with us in the resurrection. And if a person gains more knowledge and intelligence in this life through his diligence and obedience than another, he will have so much the advantage in the world to come. (D&C 130:18-19)

And though I strongly believe in and am grateful for the optimistic tone and message of those verses, I also know that the yearning sense of inadequacy inspired in me by the emptiness of deserts – the emotion the author of Ecclesiastes conveys so well – must reflect something significant. I think we've all felt it, and that alone is evidence to me that it stems from some very real part of the human condition.

That same emotion returned to me powerfully again several years later on another bright autumn afternoon. This time I was a graduate student pursuing my doctorate in mathematics at the University of Washington in Seattle. I was at the point in my degree program where only the dissertation research remained, so my afternoons were usually spent sitting at my desk scribbling on scratch paper and looking for some significant idea to break. That afternoon I chose to leave my desk and instead enjoy the sunshine outside. Taking my scratch pad and pencil with me, I walked to a quiet area of campus and settled in to work. My research project in geometry involved a technical question about tilings – the filling up of space by geometric shapes. My scratch pads would fill with patterns of tiles and formulas attempting to explain their properties. The emotion I've been trying to describe hit me that day as my eyes changed focus from the scratch pad in my hands to the leaf-tiled ground underneath me. Despite my supposed sophistication in mathematical reasoning, I was only toying with docile patterns. All around me lay complexity I could never put in formula. I picked up a single colored leaf, gazing at the intricate veining on its face, and the feeling deepened. My usual pride in thinking of mathematics as a search for pure and ultimate truth seemed shallow. Everything I studied was the faintest of shadows to an indescribable reality.

That day the way I looked at mathematics changed, and a real interest in the relationship between my faith and my scholarship began. My research interests have remained in geometry, but I've found great fascination with the philosophy inspired and informed by modern mathematics. I've been particularly impressed that some mathematics can touch in me the same chord that Ecclesiastes strikes. In particular, there is a good deal of interesting mathematics that relates directly to limits on human understanding. In this article I hope to convey some of my musings on the subject. I'll begin with two cautionary notes:

1. What I say here will involve some *speculation*, both from a *theological* and a *mathematical* viewpoint. However, what I say is consistent with current knowledge in mathematics – that is, it is at least within the realm of possibility so far as we presently know. I also believe it is similarly consistent with church doctrine in that, while it may be speculative, it contains nothing contradictory to standard church teachings.
2. The mathematics we'll need is surprisingly accessible in its general ideas, but we will need to introduce some terminology and concepts with which non-mathematicians will not be familiar. Please be patient with this, and rest assured that we won't have to deal with any actual equation-chasing or number-crunching. What I'll describe here is more *meta-mathematics* than mathematics itself, and a careful reader willing to think through the logic should be able to follow the ideas.

We'll return to the theological implications eventually. But first, there is a considerable amount of background to describe! What follows here is a quick (as quick as possible, in any case) introduction to some concepts we'll need. Some may be familiar, and some will not. In any case, I find it to be interesting material for its own sake.

Some background history

To understand *what* mathematics says about the limits of human reasoning it is nearly essential to understand *how* it is that mathematics even came to address such topics. So, we'll digress here with a thumbnail sketch of some mathematical history.

Numerical calculations were done by several cultures as early as before 3000 BC. But it was only when the Greeks introduced the notion of proof in about 600 BC that we had true mathematics. For whatever definition you choose to use for mathematics (and it is a notoriously difficult thing to define), the use of deductive reasoning to draw conclusions from a set of assumptions is at the heart of the matter. Thales (624 – 548 BC) supposedly wrote the first proofs, and by Euclid's time (about 300 BC) the Greeks had evolved the *axiomatic method*, a formalization of the deductive process in which a small set of assumptions (called *axioms*) are set forth initially and then a superstructure of proved facts (called *theorems*) is built up from that foundation. Most of us know the name Euclid from its association with geometry. But the *Elements* (the work for which he is primarily known) is most notable not because of its content but rather because of its remarkable success in use of the axiomatic method.

The success of the *Elements* helped to solidify the axiomatic method as *the* way to do mathematics. Precision in stating and tracking assumptions became the gold standard by which mathematical works are judged. Modern mathematics has taken the axiomatic method to new heights of formality, but the basic outline remains the same: begin by stating your axioms, then work carefully within the laws of logic to prove consequences of those assumptions. So fundamental is the axiomatic method to the discipline of mathematics that just as the sciences are distinguished by their use of the *scientific method*, one could characterize mathematics as the use of the axiomatic method.

The growth of mathematical understanding in the 2300 years since Euclid has been far from even. There were long periods of stagnation and even regression as well as side trips aplenty. There have also been swings in the perception of to what degree mathematics can tell us about reality. The Pythagoreans (the cult founded by Pythagoras in the 6th century BC) believed in the creed that “*all is number*” -- literally, they believed all of observable reality could be explained in the properties of the natural numbers (1, 2, 3, . . .) and their ratios. But the Greeks' faith in the rationality of the universe eventually gave way to the medieval suspicion that the universe was governed by mysterious forces operating by laws unexplainable to the human mind. And while the tool of algebra that emerged in the 13th and 14th centuries AD was put to impressive use, the applications were modest in ambition. The hope of answering big questions through mathematical reasoning would take time to reemerge.

The pendulum was moved a considerable distance toward belief in the power of reason by the invention of calculus in the late 17th century. That moment marks a turning point not just in mathematics, but in human intellectual progress. A century before Newton the prevailing world-view was laced with superstition – humans looked on the universe and its mysterious rules with suspicion. But the generations following Newton, with *Principia* in hand, saw a clockwork universe operating according to rules that were both describable and predictable. So far had the pendulum swung, that the 18th century brought an over-exuberance among mathematicians, who then worked with the hope of a complete mathematical description of everything. The train of deduction the Greeks had set in motion was back on track and running at a full head of steam.

But actually, the train had gotten a bit ahead of its own engine. Much of the voluminous work done by the great mathematicians of the 18th century was lacking in the rigor usually associated with mathematics. It was as if the mathematics community was impatient with the

slow development of rigorous methods and could not be held back from exploring the exciting new vistas opened by the methods of calculus. The work of justification could be done after the adrenaline rush.

That time came in the latter half of the 19th century. But as usually happens, the work of justification brought many difficulties and generated more questions than were actually answered. Already in the middle of that century there had been indications that the tracks our mathematical train was traveling were not headed toward a complete description of the physical universe. Non-Euclidean geometries had reared their horrifying heads as bizarre but mathematically consistent alternatives to Euclid's revered *Elements*. Mathematicians realized that the axiomatic method could be applied to many different sets of axioms, giving rise to many different mathematical universes, all of which were internally consistent, and none of which could claim to be a perfect model of physical reality. Mathematics as a whole began to chart a more independent course aimed toward abstraction rather than simple modeling of physical phenomena.

Infinity rears its head

And then there was the problematic concept of infinity. It runs all through calculus, as any freshman calculus student today can tell you. But the mathematics of the time was not equipped to deal with actual infinities, and there was a real aversion to their mention. Finally, in the late 19th century (200 years after Newton!) the German mathematician Weierstrass finally provided calculus with a rigorous base, and did so without resort to a new mathematics of infinity. But the suspicion remained that infinity would need to be conquered.

In 1874 Georg Cantor published a paper announcing the beginning of the battle for infinity. His paper had two startling implications:

- (1) there are different “sizes” of infinities.
- (2) “most” numbers are very strange.

Item (1) is a source of amazement for all who hear of it. It remains as counterintuitive today as it was when Cantor first announced it.¹ The basic idea is this: the “sizes” or *cardinalities* of two sets are compared by considering one-to-one correspondences between the sets. If set X can be put into one-to-one correspondence with a part of set Y then we say that the cardinality of Y is at least as great as that of X – that is, $|Y| \geq |X|$. (Here we use the symbol $|X|$ to denote the *cardinal number* of set X .) This certainly works for finite cardinal numbers, and in fact, is the way we intuitively learn to think of numbers as children: 3 is less than 7 because we can associate a set of three objects in a one-to-one way with only part of a set of seven objects. Cantor's breakthrough was to apply this same simple principle to *infinite* sets: if X and Y are sets with infinitely many elements each, we may still compare the sizes of X and Y by asking if X can be put into one-to-one correspondence with part of Y . If this is possible, then we can still write $|Y| \geq |X|$, just as we do in the finite case. If both $|Y| \geq |X|$ and $|X| \geq |Y|$ are true (that is, if X can be put into one-to-one correspondence with part of Y and Y can be put into one-to-one correspondence with part of X), then we conclude that $|Y| = |X|$ – the sets have equal cardinality.² However, if $|Y| \geq |X|$ holds true, but $|X| \geq |Y|$ is *not* true, then we conclude $|Y| > |X|$ -- the set Y is strictly larger (of greater cardinality) than X . Cantor managed to show that $|\mathbf{R}| > |\mathbf{N}|$ where \mathbf{R} denotes the set of real numbers and \mathbf{N} denotes the set of natural numbers $\{1, 2, 3, \dots\}$. (The real numbers \mathbf{R} include all numbers most of us ever think about – all those that can be written in decimal form, even if

¹ See the essay “To Journey Beyond Infinity” by Kent A. Bessey in *BYU Studies* 43 no. 4 (2004): 23-32 for an interesting discussion of the philosophical implications of infinity.

² There is actually a subtle mathematical twist here. We would like “equal cardinality” to mean that the two sets can be put into exact one-to-one correspondence. But $|Y| \geq |X|$ and $|X| \geq |Y|$ mean only that we can put X in one-to-one correspondence with part of Y and Y in one-to-one correspondence with part of X . Proving that these two conditions imply an exact one-to-one correspondence between X and Y was difficult enough to stump Cantor, but successful proofs were eventually produced by several mathematicians independently.

the decimal expansion never ends.) The natural numbers are obviously in one-to-one correspondence with part of the real numbers (they *are* part of the real numbers!). But Cantor proved that there can be no one-to-one correspondence between the real numbers and any set of natural numbers, so $|\mathbf{R}| > |\mathbf{N}|$. Both sets are infinite, but they are not of equal cardinality!

Mathematicians use the symbol \aleph_0 (pronounced “aleph naught” -- aleph is a character in the Hebrew alphabet) to denote $|\mathbf{N}|$, the cardinal number of the set $\mathbf{N} = \{1, 2, 3, \dots\}$. A set whose cardinal number is \aleph_0 is said to be *countable*, since putting a set in exact one-to-one correspondence with $\{1, 2, 3, \dots\}$ can be thought of as “counting” that set. But according to Cantor's work, there are cardinal numbers larger than \aleph_0 – in fact, he showed that there are infinitely many infinite cardinal numbers and that there is no largest cardinal number! Any set (such as \mathbf{R}) whose cardinal number is larger than \aleph_0 is said to be *uncountable*. Informally, countable sets are “small” infinite sets – much smaller than uncountable sets.

What will primarily interest us from Cantor's paper is item (2) above, for it is the first discovered instance of a major theme running through modern mathematics – the theme at the heart of my thesis. To understand it, we need to set out some different classes of numbers.

Types of numbers

First, we need to point out that mathematicians generally classify numbers according to the types of equations for which they might be solutions. If that seems an odd system at first, understand that centuries of experience have led to it. And actually, it does make sense; for when you get to the root of how we think of numbers, you discover that we think of them in terms of solutions to equations. If I were to ask you what $2/3$ is, you might respond that it is the result of the quantity two being divided into three equal pieces: in other words, it is a quantity x three of which would equal two -- a solution to $3x = 2$.

Now equations like this one, of the form $ax = b$ where a and b are integers, are called *linear equations*. Their solutions (the fractions a/b) are called *rational numbers* – a term you've probably heard before. Numbers that are not rational – those that cannot be written as fractions of integers – are called, of course, *irrational*. Despite the sinister sounding name, many irrational numbers are actually quite familiar to us. For instance, $\sqrt{2}$ is irrational. But while $\sqrt{2}$ may not be the solution to a linear equation, it is the solution to an only slightly more complex equation, namely $x^2 = 2$. This simple equation gives us a concrete way to think of $\sqrt{2}$, so despite being irrational it is yet fairly understandable.

Generally, a number x is said to be an *algebraic number* if it is the solution to a *polynomial equation* $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x = b$ where n is some positive integer and $a_n, a_{n-1}, \dots, a_2, a_1,$ and b are all integers. So $\sqrt{2}$, despite being irrational, is definitely algebraic. Numbers that are not algebraic are called *transcendental*. You might be more familiar with the rational/irrational split of real numbers than with the algebraic/transcendental split. But in terms of characterizing which numbers are understandable and which are not, the latter does a much better job. (Again, because we understand numbers in terms of equations – transcendental numbers are not solutions to nice equations, so in a very real sense we have no fundamental way to grasp them.) You probably can't name any transcendental numbers other than a very few famous examples like the number π (some readers may also be familiar with the number e). It isn't that you don't know enough math to know more transcendentals, it's just that most transcendentals are so bizarre in their makeup as to be beyond human description.

Now, back to Cantor. What Cantor's results proved is that the “nice” algebraic numbers, while infinite in cardinality, form a “smaller infinity” than the “messy” transcendental numbers. In fact, the algebraic numbers are countable while the transcendental numbers are uncountable.

This difference is great enough that if you choose a truly random real number, the probability that it will be algebraic is zero. Oddly (and disturbingly to Cantor's contemporaries), Cantor accomplished this proof without giving any way of actually generating transcendental numbers. In effect, his conclusion says “almost all real numbers are too strange for us to ever see or to really grasp.”

20th century mathematics gave us another division of numbers into two classes – a division that is even more fundamental to the question of what it means to “understand” a number. The ideas came from the *theory of computation* – a mathematical exploration of what computing machines can and cannot do, the groundwork of which was laid even before the development of electronic digital computers. Here, computing machines are represented simply as sets of rules for manipulating inputs into outputs. There are several such abstract models of computing machines, but the most widely accepted model is called a *Turing machine*³ (or TM for short). We won't go into any detail on what a TM is. You need only think of them as mathematical models for algorithmic processes. Any computer running any program can (in theory) be modeled by a TM. In fact, even human decision-making processes can be thought of as TMs. We say that a number x is *computable* if there exists a TM that can output x to any decimal accuracy we wish. Clearly any rational number is computable, since we can output the decimal expansion of a/b by the simple algorithmic process of long division. In fact, you probably won't have too much trouble believing that all algebraic numbers are computable – the polynomial equation that defines an algebraic number can be turned into a method for generating its decimal expansion. But the class of computable numbers is even bigger than the set of algebraic numbers since many transcendental numbers are computable. For instance, the most

³ Named for Alan Turing (1913 – 1954), one of the founders of the theory of computation, who first championed the TM concept as a model for algorithmic processes.

famous transcendental number, π , can be computed using the following striking fact from calculus:

$$\pi = 4 - 4/3 + 4/5 - 4/7 + 4/9 - 4/11 + 4/13 - 4/15 + 4/17 - 4/19 + \dots$$

Note that this doesn't mean we could ever write down all of the decimal expansion of π : we certainly can't do that, for we know that it continues forever with no repetition or apparent pattern. But if you want to know the three-millionth digit⁴ after the decimal point in π , it could be computed by this formula.⁵ Since this formula could be turned into a TM, we conclude that π is computable. Now, *any* method by which we choose to create a decimal number could be modeled by a TM. Thus, the computable numbers are the only numbers we can ever hope to “name” or write down. By *definition*, you can never write down a non-computable number!

You may have already guessed the theme of this paragraph: *most real numbers are non-computable!* In fact, the set of all TMs turns out to be a countable set (by an argument we won't go into here). But the set of computable numbers has cardinality no bigger than the set of TMs, since there is an obvious one-to-one correspondence between the computable numbers and a collection of TMs. (Namely, just associate each computable number with a TM that computes it.) So, since the real numbers are uncountable and the computable numbers are just a puny countable part of all real numbers, in a very exact way we can say that “almost all” real numbers are uncomputable, and thus beyond our comprehension.

The Law of Mathematical Unapproachability

I said above that the second-named discovery from Cantor's famous paper (the fact that we've been discussing -- that most real numbers are strange) is a precursor to a broad theme in

⁴ The three-millionth digit of π after the decimal point is a 3. In fact, digits 3,000,000 through 3,000,009 in π are 3697067915.

⁵ Actually, one really wouldn't want to use *this* particular formula, since it converges much too slowly to π . There are other similar (but more complicated) formulas that give much faster results.

modern mathematics. I call this theme the “*Law of Mathematical Unapproachability*”. It can be simply stated as “*most objects in the universe of mathematics are too wild for humans to describe.*” This statement may require some explanation for a non-mathematician to absorb. What is the “universe of mathematics”? Most mathematicians inherently believe there *is* such a universe⁶, though they would be hard-pressed to describe it to you.⁷ This universe of mathematics is as much a place to a mathematician as any physical location you've ever visited. The distinction, of course, is that mathematicians “go there” only mentally through their work. But though it may be only a work of the mind, we think of it as real nonetheless. In that universe one can find the never ending river of real numbers (with the integers scattered uniformly along it), the perfect plane of Euclid, many oceans of functions, and the mountains of infinities that build forever on themselves. But now consider that universe in the light of the Law of Mathematical Unapproachability: while I may visit the mathematical universe and tinker with a few of the pebbles I find there, most of its substance will be invisible to me. (Invisible, not undetectable – I know the non-computable numbers exist – I simply can't “see” them.) The objects mathematicians love to explore are in fact, for the most part, not within their reach.⁸

⁶ Belief in this universe of mathematical objects is central to the *Platonist* philosophy in mathematics. A Platonist mathematician believes that the mathematical objects he or she studies – the integers, the real numbers, functions, shapes, and so on – actually exist, and that their work as a mathematician consists of discovering the properties of these objects. This contrasts with the *Formalist* philosophy which holds that mathematics is a human invention, and that mathematical terms are simply abstract constructs having no real existence. Most mathematicians have a bit of both schools in them, and are quite comfortable switching back and forth between the two outlooks as occasion requires (much the way physicists become comfortable with thinking of light as particle and/or wave).

⁷ In his book *Flatterland* (Cambridge: Perseus Publishing, 2001), a modern follow-up to the classic *Flatland* by Edwin Abbot, author/mathematician Ian Stewart gives the best description I have seen to the mathematical universe (or “Mathiverse” in his terminology). See p.28-30.

⁸ Though he lived two centuries before the central ideas discussed in this paper began to emerge, Sir Isaac Newton expressed something very like this sentiment in one of his most famous quotes. Shortly before his death, he wrote in his memoirs: “*I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.*”

The predominance of the transcendental numbers (and its later extension to non-computable numbers) was merely the first proved instance of the Law of Mathematical Unapproachability. Among its many other known occurrences are the following.

- *Most continuous functions are hopelessly non-differentiable* (this is the calculus-speak way of saying that most functions have graphs that are so crinkly as to have no tangent lines anywhere). This means that our calculus applies in only a tiny corner of the universe of functions. Yet we study calculus because we can say *something* about that tiny corner, whereas we have only a few strained examples of what lies outside it.
- *Most two-dimensional shapes are fractal-like*, exhibiting infinitely complex behavior viewed at any scale. Traditional plane geometry says little about these objects, and the relatively new field of fractal geometry barely scratches the surface.
- Sets are in many ways the most fundamental objects in mathematics. Yet one of the major results in the theory of computation is that *most mathematical sets cannot be described by any TM*, including presumably the computing machines within our own skulls.

These instances of the Law of Mathematical Unapproachability are interesting, but they are of limited use in determining limits to human thought. After all, for the most part, they simply say that we can prove the existence of objects with complexity too great for human description. But, note those important words: *we can prove* . . . One could argue that to a certain degree we *do* understand non-computable numbers – we can prove they exist. We just can't write one down. The above items give us tasks we cannot perform, but not questions we cannot answer. However, there *are* questions we cannot answer. Their existence was guaranteed in one

of the most famous mathematical/philosophical developments of the 20th century: Gödel's "Incompleteness Theorems".

Gödel's Theorems: mathematics discovers its own limitations

In 1931, Kurt Gödel published his now-famous theorems on axiom systems. The exact statements of Gödel's theorems are quite technical, but we can lay out the main ideas in simple terms. Recall that an *axiom* is an assumption – something we agree to accept as true without proof. An *axiom system* is a set of such assumptions from which we hope to derive a set of useful theorems. An axiom system is said to be *inconsistent* if it is possible to prove contradictory statements from its axioms – clearly something we want to avoid. If the axioms have no such built-in contradictions then we say the axiom system is *consistent*.

Now axiom systems are somewhat stuffy and hard to think about, so let's switch over to thinking about computing machines. There's actually an easy correspondence between an axiom system and a computing machine. Imagine loading your set of axioms into a machine's memory, programming it to use correct logical inference, and then setting it to the task of outputting a list of *all* possible theorems that can be proved from those axioms⁹. If the axioms are consistent, the machine will never output two contradictory statements, so we can consider the machine to also be consistent.

In the early 20th century there were high hopes that all of mathematics (and perhaps all of the sciences as well) would eventually be axiomatized. If that happened, and if this hypothetical computing machine were constructed to work with those axioms, there would be no more need

⁹ Though it sounds like every geometry student's dream, it isn't difficult to describe – in theory anyway – how such a machine would work. It would begin by outputting all conclusions reached by "one-step proofs" -- conclusions reached by quoting one axiom. Then by beginning with these statements and following them with each of the system's axioms in turn, it can list all of the conclusions reached by "two-step proofs". These would then allow easy computation of the conclusions of "three-step proofs", and so on. Continuing in this way, any theorem that can be proved from the axioms would eventually be output.

for mathematicians. If you had a math question, you'd simply ask UMTG (the Universal Math Theorem Generator). But then why stop at math? If the sciences are also axiomatized (and human behavior and aesthetics along with them) we could build UEO (the Universal Everything Oracle) that could predict all events, write the elusive perfect novel, and in short, leave nothing for us to do.¹⁰

Fortunately for all of us, this will never happen, for Gödel's first theorem says that no such machine is possible. In fact, no consistent machine can generate all theorems in just the limited area of arithmetic of the natural numbers. No matter what axioms you build into your machine, either it will be inconsistent or there will be correct statements about arithmetic that the machine can never derive.

The idea to Gödel's proof is surprisingly simple. Imagine that we have a set of axioms, call it A , and from it we build a machine $M(A)$ that we claim is a UMTG. Gödel can prove us wrong by constructing a true statement in arithmetic that our machine will never prove. He does this by asking to see how our machine works (that is, he asks to see our axioms A), and from this he produces an arithmetic statement S that (in a complicated but very exact way) encodes the self-referential sentence "*The machine $M(A)$ will never prove this statement to be true*". Now, think about that sentence for a minute:

- If our machine proves Gödel's arithmetic statement S , it has essentially proved the above sentence to be true, which of course makes the sentence (and thus also the associated arithmetic statement S) *false*! In this case, our machine is inconsistent since it has proved a false statement.

¹⁰ This section follows ideas from chapter 4 of Rudy Rucker's excellent book *Infinity and the Mind: the Science and Philosophy of the Infinite* (Princeton: Princeton University Press, 1995).

- On the other hand, our machine certainly can't prove S to be false, for the minute it does then it has proved Gödel's sentence false – but this makes Gödel's sentence *true*! So then S is also true, and our machine is again inconsistent, having proved a true statement to be false.
- Thus, if our machine is consistent, the only possibility is that it will say nothing about statement S. But that makes Gödel's sentence true (think about it!) and thus Gödel's arithmetized version S is a correct arithmetic statement. So we then have an example of a correct arithmetic fact that our machine cannot prove. This proves Gödel's first theorem.

Gödel's second theorem is similar, but with a slight twist. It says that one thing a consistent axiom system (or computing machine, if you prefer) can never prove is its own consistency. That's a nice bit of logical irony – no consistent computing machine I design can ever prove the statement “*This machine is consistent.*” In fact, there are only two possibilities for the status of a TM equipped to do arithmetic: either it will be inconsistent (and thus useless) or it will be unable to demonstrate that it is consistent. Consider what that means for today's mathematics. We do, in fact, have a set of axioms we use as the basis of arithmetic. Gödel's second theorem says that either

- (1) those axioms are inconsistent – flawed by self-contradiction – or
- (2) we'll never know they aren't.

Those are the only two possibilities. And, of course, the only way we could ever find out which possibility actually happens is for things to go just as we don't want! It's altogether within the realm of possibility that we could wake up tomorrow to the news that someone somewhere has

discovered a contradiction in arithmetic, proving possibility (1) true. This would be disastrous! Pretty much all of mathematics rests on the properties of the real numbers, so if arithmetic goes, the whole castle comes down. (And if mathematics crumbles, what science would remain standing?) The best we can hope for is that (2) is the correct possibility – we can't *prove* it, so we have to hope for it. It's a matter of faith¹¹ and simple pragmatism. Mathematicians act on the assumption that our axioms *must* be consistent, though thanks to Gödel, we know we can never be certain.

To summarize, then, Gödel's theorems tell us two things about the limitations of mathematics:

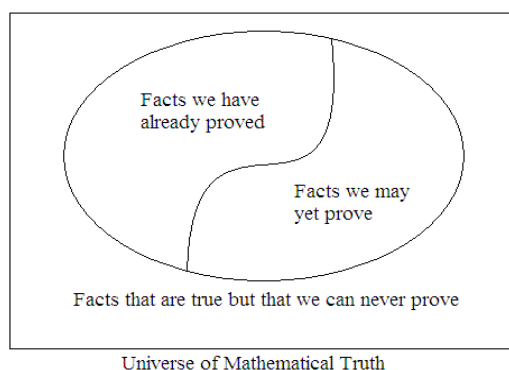
- We can never discover *all* correct mathematical facts.
- We can never be certain that the mathematics we are doing is free of contradictions.

Mathematicians have grown more-or-less accustomed to these. Most of us ignore the second one, since it's a matter of faith, and there's nothing much we can do about it. The first one intrigues us because mathematicians love unsolved problems – we're happy that there is a never-ending supply of them. There are many examples of conjectures in current mathematics that most mathematicians believe are almost certainly true, but which seem to elude proof. Perhaps some of them are in fact *unprovable* (at least with our current axioms) – instances of Gödel's first theorem. That wouldn't bother us too much. But we are prone to thinking, consciously or not, that the unprovable facts are strange exceptions and that the ones we *can* prove are the rule. After all, we only know of a few genuinely unprovable statements, so surely (we think) there must *be* only a few of them. In short, our natural tendency as human mathematicians is to assume that

¹¹ Even the most ardently atheistic or agnostic mathematicians and scientists, then, must be practitioners of the principle of faith. Without faith in the (unprovable) consistency of our mathematics, there would not be much point to pursuing mathematical or scientific questions.

nearly all the mathematics problems we encounter have solutions within the reach of human reason. But, remember the transcendental numbers! We only know of a few, but they are in fact the rule – the algebraic numbers are the exceptions! What if the Law of Mathematical Unapproachability applies to mathematical truths?

Our place in the Universe of Truth



Consider the set of all correct mathematics theorems – the Universe of Mathematical Truth. Once we decide on some axioms to use, that universe divides naturally into three parts as illustrated in the above diagram: facts for which we already have proofs, facts that have proofs we haven't found yet, and facts (we know they exist thanks to Gödel's first theorem!) that have no proofs from our axioms but that are still nonetheless true. Might not the Law of Mathematical Unapproachability suggest that *most* things in that universe fall in the third category? Might it not be that the “unprovable” part of the universe is in fact nearly everything, with the other two regions making up only an insignificantly thin slice? I don't know if that's correct. (Even if it *is* correct, that fact itself is probably one of those unprovable statements!) I don't even know the best way to measure the meaning of “most” in this setting. But I have a gut-level suspicion that something like this is what we're up against. In fact, I suspect that this picture holds no matter

what axioms we use. Gödel tells us that no choice of axioms will eliminate the *existence* of unprovable truths. I suspect the natural extension holds: no choice of axioms can eliminate the *predominance* of unprovable truths.

Let's now make the shift away from mathematics to knowledge in general. And really, it isn't much of a shift considering the generality of the mathematics we've been discussing. I've always viewed learning in general, and mathematics in particular, as an adventure – something akin to exploring a world. The analogy of a universe of facts is not really an analogy to me. As a confirmed Platonist I believe in a universe of *all truth* – a collection of “that that is”. And I find hints of this Platonist view reflected in LDS scripture: “*truth abideth and hath no end*” we read in Doctrine and Covenants 88:66. More pointedly,

“All truth is independent in that sphere in which God has placed it, to act for itself, as all intelligence also; otherwise there is no existence.” (D&C 93:30)

In Mormon theology, truth is eternal and exists independent of our ability to detect or derive it. It is absolute.

Given that we exist within such a Universe of Truth, how do we go about finding our way within it? I have viewed mathematics as a vehicle I can use in exploring (part of) the Universe of Truth. But in fact, the vehicle I call mathematics is one we all use in our exploration. For mathematics comes down to the application of reason and deduction. TMs and axiom systems are in the end just fancy ways of describing the reasoning processes we all use. Perhaps you don't use the mathematical language that I do, and perhaps you aren't interested in the more esoteric mathematical landforms that fascinate me in the Universe of Truth, but we all use deductive reasoning as one way to reach truth. So Gödel's theorems caution us all that there are places this particular vehicle can never take us. In fact, I suspect that the Universe of Truth is a wild and

rugged land, and our deduction-driven low-clearance vehicle of conscious human thought can take us to only an insignificant part of it.

The idea that there are truths beyond our reach would not surprise anybody. However, most of us are probably prone to thinking of this as a limitation of *volume* rather than of *substance*. Few would argue that there is more information out there than our minds can possibly hold. But the mathematics we've outlined suggests an awesome depth to the picture: there is truth (perhaps *most* truth, perhaps even *almost all* truth) that is of an essence and nature beyond our ability to consciously comprehend.

For what if the Law of Mathematical Unapproachability is indeed valid, and is furthermore only a shadow of the larger picture of our our position in the Universe of Truth? In that case, the knowledge we are able to obtain through our conscious reasoning would be as sparse in the true substance of truth as the computable numbers are sparse in the real numbers. Almost all objects in the Universe of Truth would defy description or approach by our puny intellects. We might think of labeling bits of truth as either “logical” (capable of being deduced by linear reasoning) or “beyond logic”. If my suspicion is correct, almost everything in the Universe of Truth is of the “beyond logic” category, but the few scattered “logical” bits are mostly what we can see. Of course, the greatest truths – the most precious gems in that universe – are probably of the “beyond logic” category. (This would give new meaning to the familiar phrase “It's only logical”.)

All of this runs counter to the prevailing western tendency to believe in the inevitable ultimate triumph of the human intellect (the same tendency that led the mathematicians in Newton's wake to assume all things would become predictable through calculus). Perhaps, though, it explains the source of the haunting feeling of inadequacy we sometimes get – the one I

associate with reading Ecclesiastes, being surrounded by desert mountains, or looking into a star-filled night sky. Perhaps our spirits sense how limited our vision truly is.

In fact, there is reason to suspect such a spiritual ability. For certainly God has access to the totality of the Universe of Truth. His ways, higher than ours as the heavens are higher than the earth, allow Him to see what we cannot. Through what means does He do this? Though God's ability to use reason and deduction would certainly exceed our own (again, heaven and earth is no doubt an apropos analogy), Gödel's theorems place limits on what can be obtained through *any* deductive process, whether that deduction is being performed by man, machine, or even God. God must have access to truth through some greater non-deductive means. I suspect that our reasoning and logic are but a shadow of a greater spiritual sense for truth – one that we glimpse here through our personal testimonies. Truth, independent in its sphere, is garnered through spiritual means in greater measure than the trickle we obtain through our linear reasoning. Indeed, most truth is inaccessible to deduction and can *only* be obtained through this greater means. Even now we can “know” far more than we can give reason for. With God, truth simply is. It needs no derivation. So it will be one day for us.

This helps me make sense of the passage (already quoted above) regarding the relationship between our efforts to learn in this life and our ability to acquire truth in the hereafter (see D&C 130:18-19). “Intelligence”, as used in Mormon scripture, is a word that obviously has profound meaning. But given the sparse explanation it receives in the scriptures (see D&C 93:29 and 36), its meaning must be difficult for us to grasp. Perhaps it is in some way a measure of our ability to obtain truth. Or perhaps intelligence *is* this greater means for truth-gathering – a means not bound by Gödel's limitations on deductive reasoning. Perhaps it is the very means by which God knows truth. The “principle of intelligence we attain unto in this life”

is what will rise with us – the next life will be a continuation of the search for truth we should be engaged in here.

Of course, in the end, I have no firm final answers. I can only speculate on the meaning of what I felt that day reading Ecclesiastes. Those of us who work in science or mathematics develop very rigid ideas of what “knowing” something means. I “know” many things from the mathematics that I have studied. *A continuous one-to-one function from a compact topological space to a Hausdorff topological space has a continuous inverse.* I know this, and I love knowing such things. What humans have achieved through deduction is both beautiful and amazing to me. In a different way I “know” that what I see here is only a dim shadow of what must really be – “*through a glass, darkly*” as Paul puts it (I Cor. 13:12) – but I “know” that someday I will see it all “*face to face*”. The first kind of knowing I can explain. It is that of a Turing machine, and given enough time and paper, I could transmit this knowing to you. Not so with the latter type. I cannot explain it even to myself. It is unearthly and mysterious. It is the distant land faintly visible to my spirit from here on the shoreline where conscious deduction ends. The gulf between here and there is, I believe, what gives our spirits such pause in those moments of this life when we confront it.