

Using Mathematics to Approach Big Questions

Mark J. Nielsen
Department of Mathematics
University of Idaho

I have to begin today with two confessions. Confession number one: even though this colloquium series is centered on interdisciplinary work, my own research in mathematics is not in the least bit interdisciplinary (unless you allow an interplay of two or more sub-fields within mathematics to count – for I use tools from topology, geometry, and combinatorics in what I do). In fact, if I were going to speak today on my own research, we might have to change the title of the series, and I'd probably consider changing the title of my talk to “Using Mathematics to Approach *Little* Questions.” And I certainly don't mean that to be a slap at the value of what I do. I work on questions that have challenged some of the greatest mathematicians of the last century, and I personally think they're fascinating and that the results we discover about them are truly beautiful. But I'm realistic about their significance in the overall scheme of humanity's existence. Let me give you one example:

How many points can you place in the plane so that no three of them are on a line and so that no six of them form a convex hexagon?

If you change that to a pentagon, the answer is 8, but for a hexagon the answer is unknown. We think it's probably 16, but all we really know for certain is that it's between 16 and 36. I know of no practical application for that problem. But I still think it's an interesting question because, well, because I'm a mathematician and questions like that intrigue me. While mathematics has always been extremely useful as a tool in other areas of study, mathematicians themselves have more often than not been completely unimpressed by that usefulness. They find instead an appreciation of their subject that is more aesthetic than utilitarian.

But to discuss what mathematics is and why we do it is really the subject for a different talk. Instead, today I really do hope to talk about *big* questions:

- **What is the human mind?**
- **What are the limits of human understanding?**

Obviously, I'm not going to settle those questions today. In fact, all I really aim to do is outline how mathematics can play a role in addressing them and making sense of them. I'm not an expert in these subjects, but I thought this topic would be a nice fit to the theme of the interdisciplinary colloquium series, unlike the “little” questions I concern myself with otherwise. And although I'm not an expert, the choice of topic reflects a very real fascination I've found in realizing that mathematics has a lot to say on these questions, and is likely to have a lot more to say in the future. That's something I would not have guessed when I began to study math some twenty years ago.

Now, I said I have two confessions to make. Here is confession number two: I'm out of my element giving a talk like this, as most mathematicians would be. The typical mathematics lecture is really quite easy to prepare and give (for mathematicians, in any case) – you define the

problem you're working on, outline the method of solution, state and/or prove the principal theorem, then illustrate with as many examples as time allows. It's all black and white, which is exactly what mathematicians are comfortable with. But while mathematics is certainly central to what I'll say today, I'll need to step *outside* of mathematics for today's discussion. In fact, addressing these questions is indeed an interdisciplinary process, requiring history, philosophy, and consideration of the generalities of science – *and* opinion – something that all mathematicians have, but few are comfortable basing lectures on. So, given that level of discomfort, and given that there are certainly people present here today far more qualified than I to discuss history, philosophy, and science, I'm actually quite happy to be forced to stick to brief, broad outlines.

How has mathematics arrived at where it is today?

To begin discussing what mathematics has to say about the nature of the human mind and the limits of its understanding we need a little set up. In particular, I'd like to give a thumbnail sketch of how mathematics came to be what it is today, for understanding that story is essential to seeing how mathematics addresses the Big Questions.

Mathematical calculations were done by several cultures even before 3000 BC, but it was only when the Greeks introduced the notion of proof in about 600 BC that we had true mathematics. For whatever definition you choose to use for mathematics (and it is a notoriously difficult thing to define), the use of deductive reasoning to draw conclusions from a set of assumptions is at the heart of the matter. Thales supposedly wrote the first proofs, and by Euclid's time (a few centuries later) the Greeks had evolved the *axiomatic method*, a formalization of the deductive process in which a small set of assumptions are set forth initially and then a superstructure of theorems built up from that foundation. So fundamental is the axiomatic method to mathematics, that just as the sciences are distinguished by their use of the *scientific method*, one could characterize mathematics as the use of the axiomatic method.

And so the train of mathematical deduction was set in motion. Let's now jump forward on its tracks almost 2000 years from Euclid to the time of Newton and the invention of calculus. I think we do a poor job today of conveying to students what a remarkable achievement calculus is. We teach it (knowing that most students will hate the subject matter) almost as if we're ashamed of it. We (and by this "we" I mean mathematics educators in general and popular calculus textbook authors in particular) give little of the sense that calculus belongs high on the list of the greatest of all human intellectual achievements, and that the development of calculus marks a turning point not just in mathematics, but in human intellectual progress.

Consider the effect that calculus had on the way humans perceived their universe. A century before Newton we find a prevailing world-view laced with superstition -- the universe viewed suspiciously with mysterious rules and workings beyond human description. The generations following Newton, with *Principia* in hand, saw a clockwork universe operating according to rules that were both describable and predictable. So far had the pendulum swung, that the 18th century brought an over-exuberance among mathematicians, who then worked with the hope of a complete description of everything. That train of deduction the Greeks had set in motion was running at a full head of steam.

But actually, the train had gotten a bit ahead of its own engine. Much of the voluminous work done by the great mathematicians of the 18th century was lacking in the rigor usually associated with mathematics. It was as if the mathematics community was impatient with the slow development of rigorous methods and could not be held back from exploring the exciting

new vistas opened by the methods of calculus. The work of justification could be done after the adrenaline rush. That time came in the latter half of the 19th century.

But there were difficulties with the work. Already in the middle of that century there had been indications that the tracks our mathematical train was traveling were not headed toward a complete description of the physical universe. Non-Euclidean geometries had reared their horrifying heads as bizarre but mathematically consistent alternatives to Euclid's revered *Elements*. Mathematics as a whole began to chart a more independent course aimed toward abstraction rather than simple modeling of physical phenomena.

Infinity rears its head: the “law of mathematical unapproachability”

And then there was the problematic concept of infinity. It runs all through calculus, as any freshman calculus student today can tell you. But the mathematics of the time was not equipped to deal with actual infinities, and there was a real aversion to their mention. The lapse was apparent from the earliest days following Newton, with critics of the new method honing in on the inadequacies of describing the vanishing “delta x ” necessary to make calculus work. Bishop George Berkeley, a clergyman with a solid distrust of the new clockwork universe (and apparently a solid enough knowledge of mathematics to write intelligently on the matter) was one such critic. He chided the champions of the new method for their use of things that are “neither finite quantities, nor quantities infinitely small, nor yet nothing,” adding that perhaps they could be termed “the ghosts of departed quantities” and that those who would be at ease with such things “need not, methinks, be squeamish about about any point in divinity.”

The epsilon-delta methods developed by Weierstrass in the late 1800's finally did provide calculus with a rigorous base, and did so without resort to a new mathematics of infinity. But the suspicion remained that infinity would need to be conquered. In 1874 Georg Cantor published a paper announcing the engagement of battle. His paper contained several startling results:

- there are different “sizes” of infinities..
- the real number line contains a greater infinity of points than does the set of rational numbers.
- most real numbers are very strange.

Let me expand on the meaning of that last item, for it is the first discovered instance of what became a major theme running through modern mathematics. Every real number can be placed in one of two categories. The *algebraic* numbers are those that are solutions to some polynomial equation with integer coefficients, such as $x^2 - 2 = 0$. Since the square root of 2 is a solution to this equation, it is an algebraic number. Numbers that are not algebraic are called *transcendental*. So transcendental numbers are not solutions to any “nice” equations. You probably can't name any transcendental numbers other than a very few famous examples like pi or e . It isn't that you don't know enough math to know more transcendentals, its just that most transcendentals are so bizarre in their makeup as to be beyond human description. Cantor proved that the “nice” algebraic numbers, while infinite in number, form a “smaller infinity” than the “messy” transcendental numbers. In effect, the algebraic numbers are so few relative to the transcendentals that if you choose a truly random real number, the probability that it will be algebraic is zero. Oddly (and disturbingly to Cantor's contemporaries), Cantor accomplished this

proof without giving any way of actually generating transcendental numbers. In effect, his conclusion says “almost all real numbers are too strange for us to ever see or to really grasp.”

I said that this fact about real numbers was a precursor to a broad theme in modern mathematics. I call this theme the “***Law of mathematical unapproachability***”. It can be simply stated as “*most objects in the mathematical universe are too wild for humans to describe.*” The predominance of the transcendental numbers was merely this law's first proved instance. Among its many other occurrences are the following.

- *Most continuous functions are hopelessly non-differentiable* (meaning that their graphs are so crinkly that they have no tangent lines). This means that our calculus applies in only a tiny corner of the universe of functions. Yet we study calculus because we can say *something* about that tiny corner, whereas we have only a few strained examples of what lies outside it.
- *Most two-dimensional shapes are fractal-like*, exhibiting infinitely complex behavior viewed at any scale. Traditional plane geometry says little about these objects, and the relatively new field of fractal geometry barely scratches the surface.
- In the theory of computation we use abstract models of computing machines to study the complexity of languages they can process. One of the major results in the subject is that *most abstract languages are beyond the reach of any computing machine*, including presumably the computing machines within our own skulls.

We'll see this *law of mathematical unapproachability* again shortly. But meanwhile, let's turn our attention back to Cantor. The mathematics community reacted harshly to his published findings, with many ridiculing the new theory of infinite sets (while of course offering no criticisms on valid mathematical grounds). It was decades before Cantor's ideas won wide acceptance. And perhaps the fears behind that hesitancy were well founded, for when the new set theory was put to exploration in the early 20th century, it was found to lead to numerous logical paradoxes. Cantor's work had exposed cracks in the very foundation of mathematics. But things would only get worse.

Gödel's Theorems: mathematics discovers its own limitations

In 1931, Kurt Gödel published his now-famous theorems on axiom systems. The exact statements of Gödel's theorems are quite technical, but we can lay out the main ideas in simple terms. Recall that an *axiom* is an assumption – something we agree to accept as true without proof. An *axiom system* is a set of such assumptions from which we hope to derive a set of useful theorems. An axiom system is said to be *inconsistent* if it is possible to prove contradictory statements from its axioms – clearly something we want to avoid. If the axioms have no such built-in contradictions then we say the axiom system is *consistent*.

Now axiom systems are somewhat stuffy and hard to think about, so let's switch over to thinking about computing machines. There's actually an easy correspondence between an axiom system and a computing machine. Imagine loading your set of axioms into a machine's memory, programming it to use correct logical inference, and then setting it to the task of outputting a list of *all* possible theorems that can be proved from those axioms. (It sounds like every geometry

student's dream!) If the axioms are consistent, the machine will never output two contradictory statements, so we can consider the machine to also be consistent.

In the early 20th century there were high hopes that all of mathematics (and perhaps all of the sciences as well) would eventually be axiomatized. If that happened, and if this hypothetical computing machine were constructed to work with those axioms, there would be no more need for mathematicians. If you had a math question, you'd simply ask UMTG (the Universal Math Theorem Generator). But then why stop at math? If the sciences are also axiomatized (and human behavior and aesthetics along with them) we could build UEO (the Universal Everything Oracle) that could predict all events, write the elusive perfect novel, and in short, leave nothing for us to do.¹

Fortunately for all of us, this will never happen, for Gödel's first theorem says that no such machine is possible. In fact, no machine can generate all theorems in just the limited area of arithmetic of the natural numbers. No matter what axioms you build into your machine, either it will be inconsistent or there will be correct statements about arithmetic that the machine can never derive.

The idea to Gödel's proof is surprisingly simple. Imagine that we have a set of axioms A and from it we build a machine $M(A)$ that we claim is a UMTG. Gödel can prove us wrong by constructing a true statement in arithmetic that our machine will never prove. He does this by asking to see how our machine works (that is, he asks to see our axioms A), and from this he produces an arithmetic statement that essentially encodes the sentence "*The machine $M(A)$ will never prove this statement to be true.*" Now, think about that sentence for a minute:

- If our machine proves Gödel's arithmetic statement, it has essentially proved Gödel's sentence to be true, which of course makes Gödel's sentence *false*!
- On the other hand, our machine certainly can't prove that sentence to be false, for the minute it does then the sentence becomes *true* (unless our machine is inconsistent and can prove it both true and false).

But since our machine never will, in fact, prove Gödel's sentence to be true, it *is* a true statement, and thus Gödel's arithmetized version of it is a correct arithmetic statement. So, our machine can't prove all arithmetic facts.

Gödel's second theorem is similar, but with a slight twist. It says that one thing a consistent axiom system (or computing machine, if you prefer) can never prove is its own consistency. That's a nice bit of logical irony – any computing machine I design can never prove the statement "*This machine is consistent.*" Consider what that means for today's mathematics. We do, in fact, have a set of axioms we use as the basis of arithmetic. Gödel's second theorem says we can never be sure those axioms aren't filled with self-contradiction. They may be perfectly consistent; mathematicians believe as a matter of faith and simple pragmatism that they *must* be consistent. But thanks to Gödel we know we can never be certain. It's altogether within the realm of possibility that we could wake up tomorrow to the news that someone somewhere has discovered a contradiction in arithmetic. Would we care? Pretty much all of mathematics rests on the properties of the real numbers, so if arithmetic goes, the whole castle comes down.

¹ This section follows ideas from chapter 4 of Rudy Rucker's book *Infinity and the Mind* (1995, Princeton University Press).

To summarize, then, Gödel's theorems tell us two things about the limitations of mathematics:

- We can never discover *all* correct mathematical facts.
- We can never be certain that the mathematics we are doing is free of contradictions.

Mathematicians have grown more-or-less accustomed to these. Most of us ignore the second one, since there's nothing we can do about it. The first one intrigues us because mathematicians love unsolved problems, so we're happy to hear that there is a never-ending supply of them. There are many examples of conjectures in current mathematics that most mathematicians believe are almost certainly true, but which seem to elude proof. Perhaps some of them are in fact *unprovable* (at least with our current axioms) – instances of Gödel's first theorem. That wouldn't bother us too much. But we are prone to thinking, consciously or not, that the unprovable facts are strange exceptions and that the types of facts we *can* prove are the rule. After all, we only know of a few genuinely unprovable statements, so surely there must *be* only a few of them.

But, remember the transcendental numbers! We only know of a few, but they are in fact the rule – the numbers we can actually grasp in our puny minds are the exceptions! What if the *law of mathematical unapproachability* applies to mathematical truths?

Imagine a universe made up of correct mathematical facts. Some of them have proofs we know about, some have proofs we just haven't discovered yet, and some (according to Gödel) have no proofs at all that we can give. Might not our unapproachability law suggest that *most* things in that universe fall in the third category? I don't know whether or not that's correct. (Even if it *is* correct, it's probably one of those unprovable statements!) I don't even know what the correct way to measure the meaning of the word “most” is in this setting. But I have a gut-level suspicion that something like this is what we're up against.

This, finally, is where I get to start talking about how I think mathematics has come to the point of addressing Big Questions. I've always viewed learning as an adventure – something akin to exploring a world. The analogy I used a moment ago about a universe of facts is not really an analogy to me. As a confirmed mathematical Platonist, it's something I believe really exists. And I've always viewed mathematics as a vehicle I can use in exploring that universe. But Gödel's Theorems tell me that there are parts of that universe my vehicle can never take me to. In fact, if my suspicion about the law of mathematical unapproachability is correct, then the mathematical universe is a wild and rugged land, and my deduction-driven low-clearance vehicle can take me to only an insignificant part of it. And there's really no reason this analogy should be limited to mathematics.

Big Question 1: What is the human mind?

Let's think about what the mathematician's search for mathematical truth can say about our own individual quests for learning. When I think of my quest to gain mathematical knowledge I have no trouble thinking of myself as a computing machine trying to spit out the next theorem derivable from my axioms – for mathematics is the application of the axiomatic method, which is modeled nicely by these machines. But really, we're all computing machines in a sense, with outputs equal to the facts we're able to deduce. (We'll ignore for the sake of this

discussion the nearly-certain conjecture that all of us believe at least one piece of utter nonsense and are thus in a way *inconsistent* computing machines!)

The formal study of computing machines was undertaken in the 20th century even before digital computers became a widespread reality of life. Alan Turing developed an abstract model of computation now called a Turing machine. His objective was to give a formal description of all deductive algorithmic processes. Turing believed that this model would apply to the human brain as well as to the PC on your desk.

There's a healthy debate about how close that model can come to the actual brain. Some would say the brain *is* a Turing machine and that all of our thoughts and conclusions are the result of its chemical and electrical workings. This is the *mechanist* point of view. The *dualists* on the other hand argue that the physical brain may (or may not) be a Turing machine, but in any case *mind* and the *brain* are not equivalent. They argue the workings of the mind cannot be explained by the Turing model – that the brain can do more than a mechanical machine can provably do. Gödel himself took a somewhat mystical dualist view, and believed his theorems provided some evidence for it.

In 1961 J.R. Lucas put forth an argument against mechanism based on Gödel's first theorem. The outline of his argument is this:

- The mechanist brings to Lucas a Turing machine M they believe is equivalent to Lucas' mind.
- Lucas produces Gödel's unprovable statement. (“M *will never prove this statement true.*”)
- Lucas knows his sentence is true, but M can never know that, so M is not equivalent to Lucas' mind.

This seems convincing on the surface, but actually can be defeated on mathematical grounds. For in order for Lucas to give the formal arithmetized version of his sentence, he would have to know the workings of M. But if M really were equivalent to his own mind, he could never do this.²

In fact, the question of what the mind really is cannot be answered with certainty by any such mathematical arguments. But the fact that it can be meaningfully explored by mathematics is impressive to me.

And there is more than just Turing machines to the mathematical exploration of what constitutes the human mind. Another model is the *neural network*, a computation structure that imitates human learning. A neural network will generate completely nonsensical answers at first, but will improve its performance over time through feedback mechanisms. In the end it may evolve a highly effective method for computing correct answers, but it will not be able to either explain its method or justify why it produces correct results. This seems to be a much better way to explain some parts of the brain's workings, giving a rationale for things like opinions and hunches.

Moreover, a neural network is not bound by Gödel's theorem. Since it is not concerned with providing proofs for its outputs, a neural network may well discover things beyond the reach of deduction. If portions of our brains work on something like this model then we may be able to “know” things that we cannot ever prove.

Big Question 2: What are the limits of human understanding?

So now imagine yourself surrounded by not just a physical universe but also an ideal universe consisting of facts to discover and understand. Here, let's take “*understand*” to mean something beyond the hunches that a neural network might provide. To understand a fact is to be

² See chapter 6 of Donald Gillies' *Artificial Intelligence and Scientific Method* (1996, Oxford University Press).

able to explain it – to have a Turing machine-like derivation of it. So just as Gödel's first theorem implies I can never derive all truths in the mathematical universe, it also means that none of us can “understand” everything in this larger universe of all truth.

And it isn't simply a matter of *volume*, for surely none of us would be surprised to learn that there's too much out there for our minds to hold. The problem, rather, is one of *essence*. The lesson of Gödel's theorems is that there are truths whose very nature is, in an exact, definable sense, beyond our grasp. This runs counter to the prevailing western tendency to believe in the inevitable ultimate triumph of the human intellect. That same tendency led the mathematicians in Newton's wake to assume all things would become predictable to mathematics. But of course, it was not to be so.

I believe we'll eventually discover that the law of mathematical unapproachability does in some sense apply to the universe of all truth – that the things in that universe we can “understand” make up the slenderest of subsets. But despite having just completed a century of discoveries like Gödel's theorems in mathematics and the uncertainty principle in physics, there is still an undercurrent of belief that the only obstacle to us understanding the underlying structure of the universe is that there may *be* no underlying structure. I'd like to spend the remainder of my time addressing that issue.

Rule vs. randomness.

The clockwork universe envisioned in the 18th century has certainly not played out. We now know there is no formulaic system of prediction for everything. One possible explanation for this is the introduction of randomness to the workings of our universe. In some ways this is a comforting explanation – we can't give a predictable rule for what happens because there *is* no rule. And certainly randomness may play a role in how things actually do work. But I don't believe it is the only possible explanation.

Some of the mathematics evolving in recent decades shows that the simple concept (or so we assumed!) of *pattern* may possess an ability to explain complex behavior that we did not previously suspect. In 1970 Princeton mathematician John Conway introduced his “*Game of Life*” -- a simple use of things called cellular automata that exhibits amazingly complex behavior from simple rules and patterns. The “game” is played on a grid of squares, each regarded as a cell that can be in one of two states, alive or dead. The living cells in one generation determine the next generation by the following rules:

- A living cell with fewer than two living neighbors will die of loneliness.
- A living cell with more than three living neighbors will die of overcrowding.
- A cell with either two or three living neighbors will go on living.
- A dead cell that has exactly three living neighbors will spring to life.

With these rules, rather simple initial arrangements of cells can exhibit complex interesting behavior over time, giving rise to patterns that grow, move across the grid, and even give birth to offspring.

A similar game can be played on the square grid by regarding each row as a generation and letting that row determine the configuration of the row beneath it. (So, time is regarded as moving downward through the grid.) In this case, each cell in the next generation has three cells from the current generation that might be called its “parents” -- a left parent, center parent, and right parent. Those three parents have eight possible live/dead configurations (see Figure 1). To create a rule for generating successive generations we simply choose which of the eight parental

configurations will have live cell offspring. We then begin with a first generation consisting of a single live cell, and see what happens.

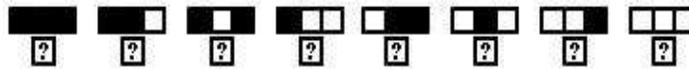


Figure 1. The 8 possible live/dead arrangements for the parents of a cell.

Many of the possible sets of rules give rise to completely uninteresting patterns. Some, like the rule shown in Figure 2, generate interesting patterns, but the patterns show predictable and describable behavior³.

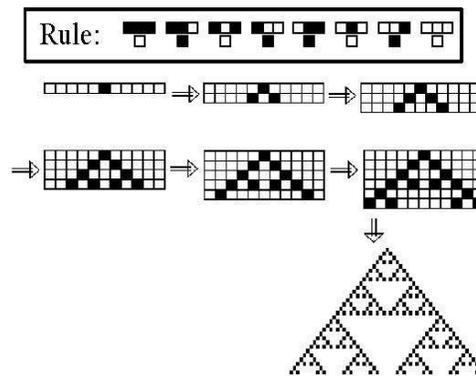


Figure 2. A rule giving a predictable pattern.

However, some rules generate patterns whose behavior seems completely “random” and indescribable. The rule of Figure 3 generates the pattern shown in Figures 4 and 5.

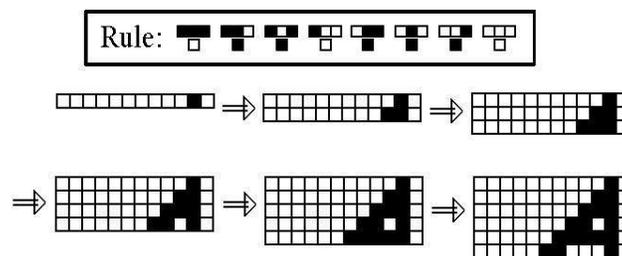


Figure 3. A rule giving rise to a complex and seemingly unpredictable pattern. See Figures 4 and 5 on the following pages.

There seems to be no way to predict what the 2000th row of this array will be short of just calculating all of the rows to that point. Even though we can completely describe the rules that govern the system, the system does not seem to be at all predicable!

³ The examples in this section are discussed in Stephen Wolfram's book *A New Kind of Science* (2002, Wolfram Media).

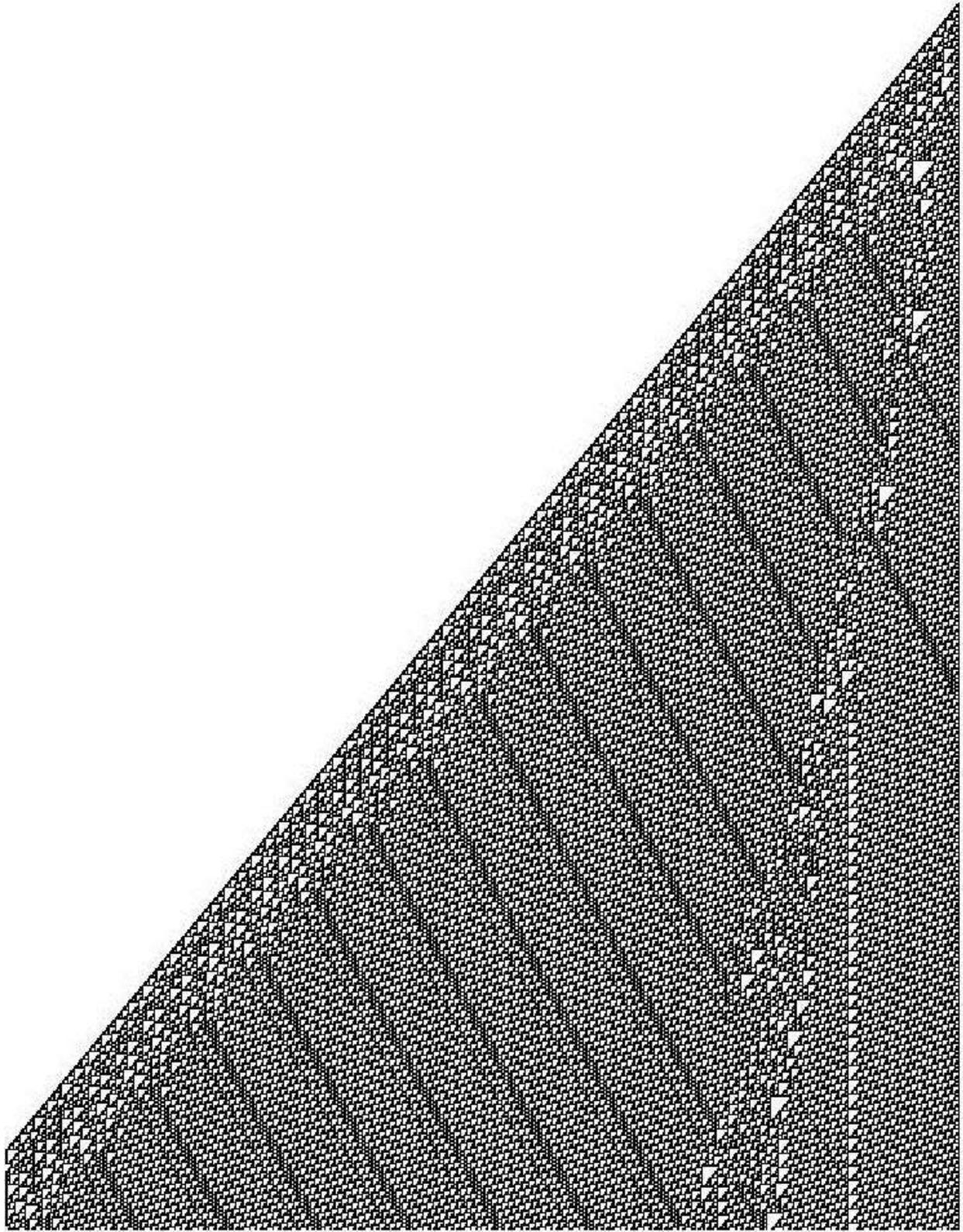


Figure 4. The first 600 rows generated by the rule in Figure 3.

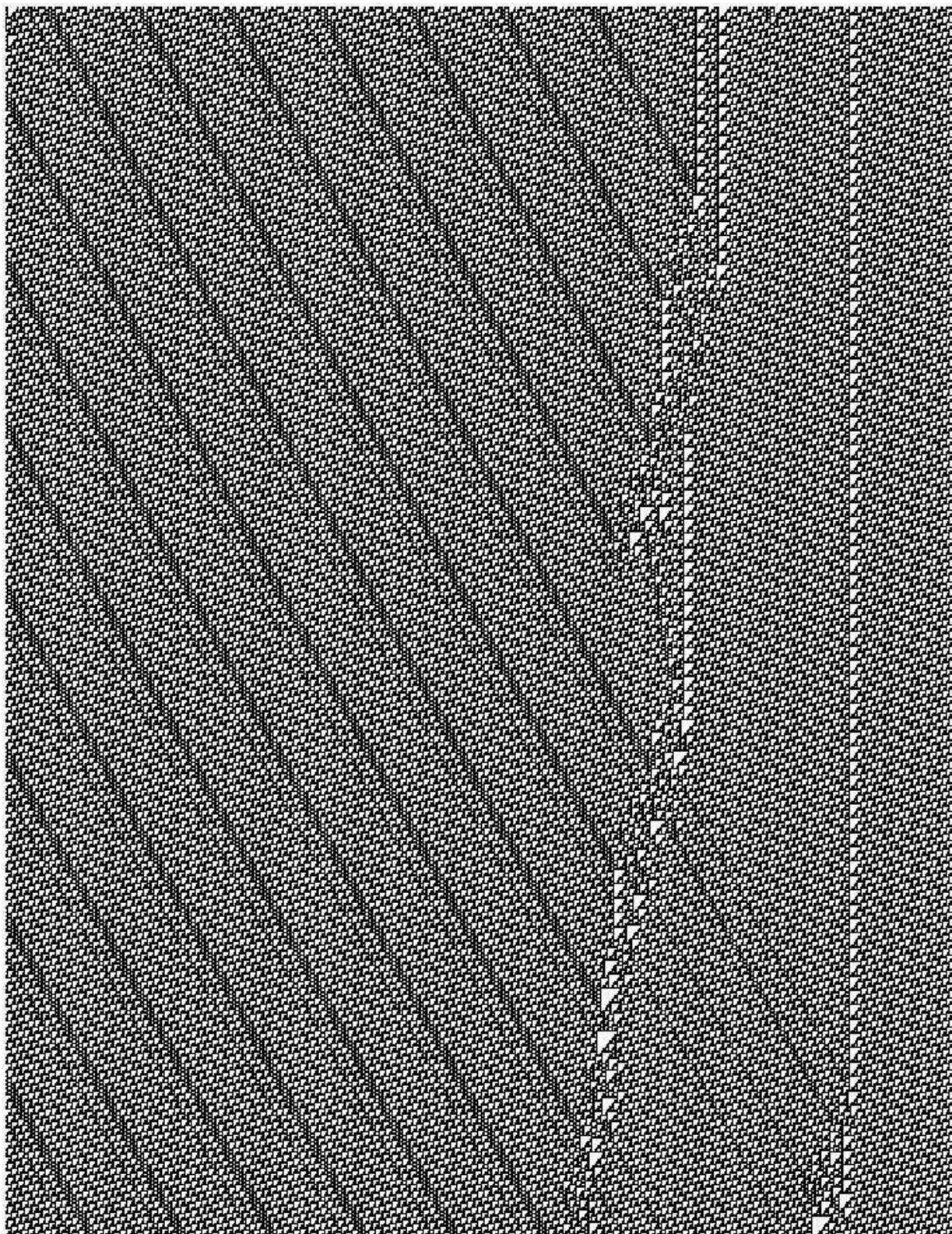


Figure 5. Portions of rows 601-1200 generated by the rule in Figure 3.

gives its location at that instant. In Figure 7, the curve in space-time corresponds to a particle moving on an arc from point A through point B to point C. The times a, b, and c when the particle is respectively at A, B, and C are shown.

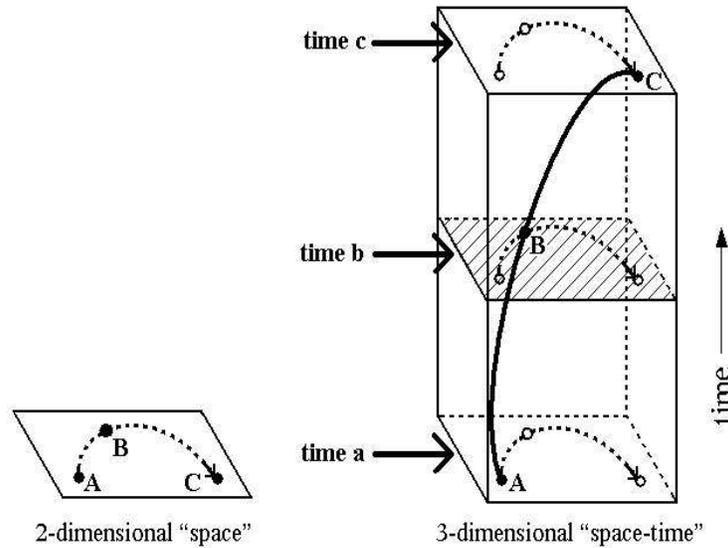


Figure 7. A curve in space-time corresponding to a particle moving along an arc from A through B to C.

Now here's the connection with the patterns we've looked at: we are made up of particles, and so, in the space-time model, we are collections of these curves. But no particle will be part of you for your entire life. They come and go, for a time traveling along with the other particles that make up your body, then going their own way. So you, or at least your physical self, appear as a persistent *pattern* in the intricate dance of these curves. This somewhat bewildering image suggests we might be something like a complex version of the moving figures in Conway's *Game of Life*. Whether simple rules govern the dance of those curves, as they do the actions of the blinking cells in Conway's game, we may never resolve. Indeed, the answer may be of the sort we *cannot* ever reach.