Homework 7

MATH 430, Fall 2014

Due: Friday, 31 October, in class.
1) All work must be shown clearly for full credit. You must justify all your answers.
2) You are encouraged to discuss the problems with other students but solutions must be written independently.

Section 6.1

1. In \( C([0, 1]) \) (the space of continuous functions defined on \([0, 1]\)), define inner product by
   \[
   \langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.
   \]
   Let \( f(t) = t \) and \( g(t) = e^t \). Compute \( \langle f, g \rangle \), \( \|f\| \), \( \|g\| \) and \( \|f + g\| \). Verify both the Cauchy-Schwarz and the triangle inequality.

2. (Pythagorean Theorem) Let \( V \) be an inner product space, and suppose that \( x \) and \( y \) are orthogonal vectors in \( V \). Prove that \( \|x + y\|^2 = \|x\|^2 + \|y\|^2 \).

3. Let \( \{v_1, v_2, \ldots, v_k\} \) be an orthogonal set in \( V \), and let \( a_1, a_2, \ldots, a_k \) be scalars. Prove that
   \[
   \left\| \sum_{i=1}^k a_i v_i \right\|^2 = \sum_{i=1}^k |a_i|^2 \|v_i\|^2.
   \]

Section 6.2

4. In the following, apply the Gram-Schmidt process to the given subset \( S \) of the inner product space \( V \) to obtain an orthonormal basis \( \beta \) for \( \text{span}(S) \). Then express the given vector \( h \) as a linear combination of the vectors in \( \beta \).
   (a) \( V = \mathbb{R}^3 \), \( S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\} \), and \( h = (1, 1, 2) \).
   (The inner product for \( \mathbb{R}^3 \) is the usual dot product.)
   (b) \( V = P_2(\mathbb{R}) \) with inner product \( \langle f, g \rangle = \int_0^1 f(t)g(t)\,dt \), \( S = \{1, x, x^2\} \) and \( h = 1 + x \).