Applications:

1) Image processing

2) Movie recommendation by Netflix

3) Mechanics: model of a vibrating bridge

\[ \frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 \]
\[ \frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 \]
\[ \Rightarrow d x = A x \]
\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]
\[ x = (x_1, x_2)^T \]
\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]
Notation
\( \mathbb{R} \): real nos., \( \mathbb{N} \) - natural nos, \( \mathbb{Z} \) - integers
\( \mathbb{C} \): complex numbers, \( \mathbb{R}^2 \) - xy plane, \( \mathbb{R}^3 \) - xyz plane
\( \mathbb{R}^n/\mathbb{C}^n \) - n dimensional space
\( \forall \): for all, \( \in \): in/belong to, \( \exists \): there exists
\( : \), s.t. - such that, \( \implies \): implies

Study objects called "vector spaces" and transformations of vector spaces called linear transformations.

Vector - has both magnitude & direction. Examples velocity, force, acceleration.
Two vectors having the same length & direction are identical regardless of position.

The coordinates of the end point can also be used to represent a vector.

Magnitude = length of the vector
Direction = arrow

\[(a_1, a_2)\]
Addition of vectors (parallelogram law)

Scalar multiplication

For a scalar $t$:

$t > 0$, length of $tx = |t| |x|$, direction is same as $x$

t < 0, length of $tx = |t| |x|$, direction is reversed
Properties of vectors:

1) \( x + y = y + x \)  \text{ commutative}
2) \( (x+y)+z = x + (y+z) \)  \text{ associative}
3) \( x + \vec{0} = x \)  \( \vec{0} \) is the zero vector
4) For each \( x \), \( \exists y \) s.t. \( x + y = \vec{0} \)
5) For each \( x \), \( 1 \cdot x = x \)  \( 1 \) - scalar
6) \( a, b \in \mathbb{R}, \ (ab)x = a(bx) \)
7) \( a(x+y) = ax + ay \)  \text{ distributive}
8) \( (a+b)x = ax + bx \)

Many other objects satisfy these properties (not just vectors in plane):
matrices, polynomials ...

Next Class - vector spaces
A set a scalars called a field, denoted by $F$.

Think of reals $\mathbb{R}$ and complex nos. $\mathbb{C}$. 

$\xrightarrow{\text{see Appendix C.}}$
A vector space \( V \) over a field \( F \) is a set on which two operations (addition & scalar multiplication) are defined so that for \( x, y \in V \), \( x+y \in V \), for each \( a \in F \), \( ax \in V \), and the following hold:

1. \( x + y = y + x \) commutative
2. \( (x+y)+z = x + (y+z) \) associativity
3. \( \exists \vec{0} \in V \) s.t. \( x + \vec{0} = x \) for each \( x \)
4. For each \( x \in V \), \( \exists y \in V \) s.t. \( x+y = \vec{0} \)
5. For each \( x \in V \), \( 1 \cdot x = x \)
6. For each \( a, b \in F \) & each \( x \in V \): \( (ab)x = a(bx) \)
7. For each \( a \in F \) & \( x, y \in V \): \( a(x+y) = ax + ay \)
8. For \( a, b \in F \) and \( x \in V \): \( (a+b)x = ax + bx \)

Elements in \( F \) are called scalars, elements in \( V \) are called vectors.
Examples (of vector spaces):

1. $\mathbb{R}^n$ or $\mathbb{C}^n$ is a vector space.
   The elements are $n$-tuples $(a_1, a_2, \ldots, a_n)$.
   
   \[ a_1 \in \mathbb{R}/\mathbb{C} \quad \cdots \quad a_n \in \mathbb{R}/\mathbb{C} \]

   $\mathbb{R}^2$:
   \[ (a_1, a_2) \]

   $\mathbb{R}^3$:
   \[ (a_1, a_2, a_3) \]

   **Addition** (component-wise):

   \[ x = (x_1, \ldots, x_n) \]
   \[ y = (y_1, \ldots, y_n) \]
   \[ x + y = (x_1 + y_1, \ldots, x_n + y_n) \]

   **Scalar multiplication** (component-wise):

   \[ C x = (C x_1, C x_2, \ldots, C x_n) \]

   $C \in \mathbb{R}/\mathbb{C}$

   Zero vector \((0, 0, \ldots, 0)\)
2. Matrices: set of $m \times n$ matrices, denoted by $M_{m \times n}$, is a vector space.

Think of matrix addition & scalar multiplication.

Zero vector: $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

Zero matrix

3. Polynomials: $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $n \geq 0$.

$a_0, a_1, \ldots, a_n \in \mathbb{F} (\mathbb{R} \text{ or } \mathbb{C})$

Coefficients.

If $a_n \neq 0$, degree = $n$. 
$P(F)$, the set of all polynomials with coefficients from the field $F$, is a vector space.

Addition:
$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$, $a_n \neq 0$
$g(x) = b_0 + b_1 x + \cdots + b_{m-1} x^{m-1} + b_m x^m$, $b_m \neq 0$

Let $m \leq n$. Think of $g(x)$ as
$g(x) = b_0 + b_1 x + \cdots + b_{m-1} x^{m-1} + b_m x^m + 0 x^{m+1} + \cdots + 0 x^n$

$(f+g)(x) = (a_0+b_0) + (b_1+a_1) x + \cdots + (a_{m}+b_{m}) x^{m} + a_{m+1} x^{m+1} + \cdots + a_n x^n$

$\deg (f+g) = \max (\deg f, \deg g) = \max (m, n)$

Scalar multiplication:
$c f(x) = c a_0 + c a_1 x + \cdots + c a_{n-1} x^{n-1} + c a_n x^n$

Zero vector: zero polynomial (all coeffs. zero), $f_0(x) = 0$.

Degree is defined to be $-1$. 
Example of a set that is \textit{not} a vector space:

\[ S = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\} = \mathbb{R}^2. \]

Consider \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \in S \)

and define

\[ x + y = (x_1 + y_1, x_2 - y_2) \]

\[ c x = (c x_1, c x_2) \]

\[ y + x = (x_1 + y_1, y_2 - x_2) \]

\[ (a + b) x = ((a + b) x_1, (a + b) x_2) \]

\[ a x + b x = (a x_1, a x_2) + (b x_1, b x_2) = (a x_1 + b x_1, a x_2 - b x_2) \]

\[ = ((a + b) x_1, (a - b) x_2) \]
Fact: The zero vector is unique.

Thm (Cancellation Law of addition)
If $x, y, z \in V$ s.t. $x + z = y + z$ then $x = y$. 
Last lecture: Vector spaces

Today: Subspaces

A subset $W$ of a vector space $V$ (over a field $F$) is called a subspace if it forms a vector space in its own right.
A subset \( W \) of a vector space \( V \) is a subspace if and only if:

(a) \( \vec{0} \in W \)

(b) If \( x, y \in W \), \( x+y \in W \) [closed under addition]

(c) For every \( c \in F \) and \( x \in W \), \( cx \in W \)

Example 1: \( V = \mathbb{R}^3 = \{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R} \} \)

\( W = \{ (x_1, x_2, x_3) : x_1 = 0 \} \)

\( W \) is a subspace of \( V \).

(a) \( \vec{0} = (0, 0, 0) \in W \)

(b) \( a = (0, a_1, a_2) \), \( b = (0, b_1, b_2) \in W \)

\( a+b = (0, a_1+b_1, a_2+b_2) \in W \)
(c) \( c \in \mathbb{R}, \; c a = c(0, a_1, a_2) = (0, ca_1, ca_2) \in W \)

Similarly, \( \{ (x_1, x_2, x_3) : x_2 = 0 \} \) & \( \{ (x_1, x_2, x_3) : x_3 = 0 \} \) are also subspaces.

Example 2 \( V = M_{m \times n}, \; F = \mathbb{R} \)

\( W = m \times n \) matrices with non-negative entries.

Is \( W \) a subspace? No

Let \( A \in W \), let \( c \) be a negative scalar.

The \( cA \not\in W \), \( cA \) may have negative entries.

\( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \; \; c = -1 \; \; cA = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \not\in W \)
Example 3: Trivial examples (of subspaces):

- $W = V$ is a subspace of $V$
- $W = \{ \vec{0} \}$ is a subspace of $V$

Example 4: $V = \mathbb{R}^3$, $Y = \{ (x_1, x_2, x_3) : x_2 = 0 \}$

$\mathbb{Z} = \{ (x_1, x_2, x_3) : x_1 = 0 \}$, $Y$ & $\mathbb{Z}$ are both subspaces.

What about $Y \cup \mathbb{Z}$? No

$y = (1, 0, 3) \in Y$, $z = (0, 2, 5) \in \mathbb{Z}$

$y, z \in Y \cup \mathbb{Z}$, $y + z = (1, 2, 8) \notin Y$ or $\mathbb{Z}$

$\Rightarrow y + z \notin Y \cup \mathbb{Z}$

Thus $Y \cup \mathbb{Z}$ is NOT a subspace.
Conclusion: The union of subspaces need not be a subspace.

Theorem: If $Y, Z$ are subspaces of $V$, then $Y \cap Z$ is also a subspace of $V$.

Proof:
(a) $\bar{o} \in Y$, and $\bar{v} \in Z$, thus $\bar{o} + \bar{v} \in Y \cap Z$

(b) Take $a, b \in Y \cap Z$.
   $a, b \in Y \Rightarrow a + b \in Y$ [Y is a subspace]
   $a, b \in Z \Rightarrow a + b \in Z$ [Z is a subspace]

$\Rightarrow a + b \in Y \cap Z$
(c) Take \( x \in Y \cap Z \) and \( c \in F \).
\[ x \in Y \Rightarrow cx \in Y \quad \text{[Y is a subspace]} \]
\[ x \in Z \Rightarrow cx \in Z \quad \text{[Z is a subspace]} \]
\[ \Rightarrow cx \in Y \cap Z \]

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In general, the intersection of any number of subspaces of a vector space \( V \) is also a subspace of \( V \). This is a way of creating new subspaces from known ones.
A - a matrix, transpose of $\mathbf{A} - \mathbf{A}^t$ obtained by interchanging rows and columns

$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 9 \end{bmatrix}$ \quad $\mathbf{A}^t = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ 1 & 9 \end{bmatrix}$

If $\mathbf{A}^t = \mathbf{A}$, then $\mathbf{A}$ is said to be symmetric. Symmetric matrices must be square: $m = n$.

$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, $\mathbf{A}^t = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \mathbf{A}$
Example

\[ V = M_{n \times n} \] , \[ W \] - all symmetric matrices in \( V \)

\( W \) is a subspace of \( V \).

(a) zero matrix is symmetric, belongs to \( W \)

(b) Let \( A, B \in W \). Need to show \( A + B \in W \).

\[(A + B)^t = A^t + B^t = A + B \Rightarrow A + B \in W\]

Property of transpose

\( A^t = A \)

\( B^t = B \)

(c) Let \( c \in \mathbb{R} \), \( A \in W \). Want to show \( cA \in W \)

\[(cA)^t = cA^t = cA \Rightarrow cA \text{ is also symmetric}\]