Previous lecture: $T: V \rightarrow W$

Linear transformations: definition, range, domain & null space of a linear transformation, example

$x, y \in V$
$c \in F$

$N(T) = \{ x \in V : T(x) = 0_W \}$

$T(x+y) = T(x) + T(y)$
$T(cx) = c \cdot T(x)$

Today: some common examples of linear transformations, further properties ...
Examples (from geometry)

(a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(a_1, a_2) = (a_1, -a_2)$

Reflection about x-axis

$T$ is linear.

(b) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(a_1, a_2) = (a_1, 0)$

Projection on the x-axis

$T$ is linear.
(c) Rotation

\[ a_1 = r \cos \alpha \]
\[ a_2 = r \sin \alpha \]
\[ r^2 = a_1^2 + a_2^2 \]

\( T_\theta : \) rotation of a point (counterclockwise) by \( \theta \).

\[ T_\theta (a_1, a_2) = (r \cos (\theta + \alpha), \ r \sin (\theta + \alpha)) \]

\[ = (r \cos \theta \cos \alpha - \sin \theta \sin \alpha, \ r \cos \alpha \sin \theta + \sin \alpha \cos \theta) \]

\[ = \left( \frac{a_1 \cos \theta - a_2 \sin \theta}{a_2}, \ \frac{a_1 \sin \theta + a_2 \cos \theta}{a_2} \right) \]

\[ = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \]
Rotation by $\theta$ clockwise is given by

\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
-sin \theta & \cos \theta
\end{bmatrix}
\]

Examples from calculus

(a) Differential operator $V = W = \text{all continuous functions}$

$T : V \rightarrow W$

$f(t) \mapsto f'(t), \quad \frac{df(t)}{dt}$

$D(T) = \text{all differentiable functions}$

$R(T) = W, \quad N(T) = \text{constant functions}$

$= \text{span} \{1\}$

Differential operator is linear.
Integral operator $V = C(\mathbb{R})$

$T: V \longrightarrow \mathbb{R}$ given by $T(f) = \int_{a}^{b} f(t) \, dt$\hspace{1cm} \forall f \in V$

$T$ is linear.
Theorem: \(V\) and \(W\) are vector spaces, \(T: V \rightarrow W\) is linear. Then \(N(T)\) and \(R(T)\) are subspaces of \(V\) and \(W\), respectively.

Proof (i) \(N(T)\):

(a) Since \(T\) is linear we know that
\[
T(O_v) = O_w \Rightarrow O_v \in N(T).
\]
(b) Let \(x, y \in N(T)\). Then
\[
T(x) = O_w, \quad T(y) = O_w.
\]
Let \(T(x + y) = T(x) + T(y) = O_w + O_w = O_w\).
\(T\) is linear \(\Rightarrow x + y \in N(T)\).
(c) Let \(c \in F\), \(x \in N(T)\); \(T(x) = O_w\).
\[
T(cx) = cT(x) = c \cdot O_w = O_w.
\]
\(T\) is linear \(\Rightarrow cx \in N(T)\).
\[
\Rightarrow cx \in N(T)
\]
Thus \(N(T)\) is a subspace.
(ii) $R(T)$: (a) Since $T(0_v) = 0_w \Rightarrow 0_w \in R(T)$.

(b) Let $x, y \in R(T)$. Then $\exists u, v \in V$ s.t.

$T(u) = x$ & $T(v) = y$.

$T(u + v) = T(u) + T(v) = x + y \Rightarrow x + y \in R(T)$

linear

(c) Let $c \in F$, let $x \in R(T) \Rightarrow \exists u \in V$ s.t.

$T(u) = x$. Want to show: $c x \in R(T)$

$T(cu) = c T(u) = c x \Rightarrow c x \in R(T)$.

linear

Thus $R(T)$ is a subspace of $W$. 
Finding the range of a linear transformation

Vector space $V$'s basis is $\{v_1, v_2, \ldots, v_m\}$

Vector space $W$'s basis is $\{w_1, w_2, \ldots, w_n\}$

Linear $\leftarrow T: V \rightarrow W$

Let $x \in V$. $x = a_1v_1 + a_2v_2 + \cdots + a_mv_m$.

$T(x) = T(a_1v_1 + a_2v_2 + \cdots + a_mv_m)$

$T$ is linear $\Rightarrow a_1T(v_1) + a_2T(v_2) + \cdots + a_mT(v_m)$

$\Rightarrow T$ is completely determined by its action on the basis of $V$.

Knowing $T(v_1), T(v_2), \ldots, T(v_m) \Rightarrow$ we know $T(x)$ for any $x \in V$. 

Thm: \( T: V \rightarrow W \), \( T \) is linear. If \( \{v_1, v_2, \ldots, v_m\} \) is a basis of \( V \) then \( R(T) = \text{span} \{ T(v_1), T(v_2), \ldots, T(v_m) \} \)

**Example:**

\[
T: P_2(\mathbb{R}) \rightarrow M_{2\times2}(\mathbb{R})
\]

Find \( R(T) \) & \( N(T) \).

Basis of \( P_2(\mathbb{R}) \): \( \{1, x, x^2\} \)

\[
T(1) = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \quad T(x) = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad T(x^2) = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Eq. 10
(Sec. 2.1)
\[ T(x^2) = \begin{bmatrix} 1^2 - 2^2 & 0 \\ 0 & 0^2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ f = x^2 \]

\[ \mathbf{R}(T) = \text{span} \{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \} \]

\[ = \text{span} \{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \} \]

\( \text{dim} \ (\mathbf{R}(T)) = 2 \)

\( \text{dim} \ (\mathbf{V}) = \text{dim} \ (P_2(\mathbf{R})) = 3 \)

\( \mathbf{N}(T) = \text{all polynomials in } P_2 \text{ that are mapped to } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

\( f(x) = a_0 + a_1 x + a_2 x^2 \) is a poly. in \( P_2 \).

\[ f(0) = a_0 \]

\[ f(1) = a_0 + a_1 + a_2 \]

\[ f(2) = a_0 + 2a_1 + 4a_2 \]

\( f(0) \) must be zero \( \Rightarrow a_0 = 0 \)

Want: \( f(1) = f(2) \) \( \Rightarrow a_0 + a_1 + a_2 = a_0 + 2a_1 + 4a_2 \)

\[ a_1 = -3a_2 \]

\( f(x) = -3a_2 x + a_2 x^2 = a_2 (-3x + x^2) \) are in \( \mathbf{N}(T) \).
\[ N(T) = \text{Span} \{-3x + x^2\} \quad \text{Basis} = \{-3x + x^2\} \quad \text{dim}(N(T)) = 1. \]

In the above example:

\[ T : P_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R}) \]
\[ \text{dim}(R(T)) = 2, \quad \text{dim}(N(T)) = 1, \quad \text{dim}(P_2) = 3 \]
\[ \text{dim}(R(T)) + \text{dim}(N(T)) = \text{dim}(P_2) \]

\[ T : V \rightarrow W \]

Nullity \((T) = \text{dim}(N(T)) \]

Rank of \((T) = \text{dim}(R(T)) \]

dimension of nullspace \quad \text{dimension of range}

**Dimension Theorem:** \[ T : V \rightarrow W, \ V \text{ is finite dimensional}, \text{ then,} \]

\[ \text{nullity}(T) + \text{rank}(T) = \text{dim}(V) \]
Proof (sketch) of the dimension thm: Let \( \dim(V) = n \) and let \( \text{nullity}(T) = k \). Want to show that \( \dim(\text{R}(T)) = n - k \).

Let \( \{v_1, \ldots, v_k\} \) be a basis of \( \text{N}(T) \).

Expand this to a basis of \( V \) (since \( \text{N}(T) \subseteq V \) & \( \{v_1, \ldots, v_k\} \) is l.i.)

\( \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\} \)

Let \( S = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\} \)

Claim: \( S \) is basis of \( \text{R}(T) \).

Next steps: (a) Show that \( S \) is l.i.
(b) Show that \( \text{span}(S) = \text{R}(T) \)
Review of one-to-one, onto functions 9/19/14

A function $T : \mathcal{V} \to \mathcal{W}$ is one-to-one if for any $x_1, x_2 \in \mathcal{V}$, $T(x_1) = T(x_2)$ implies $x_1 = x_2$.

E.g. 1) $f(x) = 2x + 3$ is one-to-one.

\[ f(x_1) = f(x_2) \implies 2x_1 + 3 = 2x_2 + 3 \implies x_1 = x_2 \]

2) $f(x) = x^2$ is NOT one-to-one.

\[ x_1^2 = x_2^2 \implies x_1 = x_2 \quad x_1 = 2, \ x_2 = -2 \quad x_1 \neq x_2 \]

A function $T : \mathcal{V} \to \mathcal{W}$ is onto if $\text{R}(T) = \mathcal{W}$, i.e. for every $w \in \mathcal{W}$, $\exists v \in \mathcal{V}$ s.t. $T(v) = w$.

$f(x) = x^2$ is NOT onto, take $y = -4$, $\exists$ any $x$ s.t. $x^2 = -4$. 
Facts that will be used:

- \( \{ \overrightarrow{0} \} \) is a subspace of \( V \).
- \( \{ \overrightarrow{0} \} := \text{span} \{ \Phi \} \); \( \Phi \) is l.i. (Sec 1.5)
  \( \Rightarrow \) \( \Phi \) is a basis for \( \{ \overrightarrow{0} \} \)
- \( \dim \{ \overrightarrow{0} \} = 0 \) \( [\Phi \text{ has no elements}] \)
Example \( T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R}) \)

Define \( T \) by \( T(f(x)) = f'(x) + \int_0^x 2f(t)\,dt \) is in \( P_2 \)

\[ R(T) = \begin{cases} 2x, & 1 \end{cases} \]

Basis of \( R(T) = \{2x, 1+x^2, 2x + \frac{2x^3}{3}\} \)

\[ \dim R(T) = 3, \quad \dim(P_2) = 3, \quad \dim(N(T)) = 0 \] by the dimension thm.
\[ \dim (N(T)) = 0 \implies N(T) = \{ \vec{0} \} \]

\[ \implies T \text{ is one-one}. \]
Theorem: \( T: V \rightarrow W \) is a linear transformation. Then \( T \) is one-to-one if and only if \( \text{N}(T) = \{ \overrightarrow{0}_V \} \)

\[ \iff \]

Proof: \((\Rightarrow)\) Assume that \( T \) is 1-1. Need to show that \( \text{N}(T) = \{ \overrightarrow{0}_V \} \). Since \( T \) is linear, \( T(\overrightarrow{0}_V) = \overrightarrow{0}_W \).

Let \( x \in \text{N}(T) \). Then \( T(x) = \overrightarrow{0}_W \). Therefore \( T(\overrightarrow{0}_V) = T(x) \Rightarrow x = \overrightarrow{0}_V \) since \( T \) is 1-1.

Thus \( \text{N}(T) = \{ \overrightarrow{0}_V \} \)

\((\Leftarrow)\) Assume that \( \text{N}(T) = \{ \overrightarrow{0}_V \} \). Need to show: \( T \) is 1-1.

Let \( T(x_1) = T(x_2) \Rightarrow T(x_1) - T(x_2) = \overrightarrow{0}_W \)
\[ \Rightarrow T(x_1 - x_2) = \overrightarrow{0}_W \]
\[ \Rightarrow x_1 - x_2 \in \text{N}(T) \]
\[ \Rightarrow x_1 - x_2 = \overrightarrow{0}_V \] (since \( \text{N}(T) = \{ \overrightarrow{0}_V \} \))
\[ \Rightarrow x_1 = x_2 \Rightarrow T \text{ is 1-1} \]
Theorem: Let \( \dim(V) = \dim(W) < \infty \)

\[ T: V \rightarrow W \text{ is linear. Then the following are equivalent} \]

(a) \( T \) is one-to-one

(b) \( T \) is onto

(c) \( \text{rank}(T) = \dim(V) = \dim(\text{R}(T)) \)