Step 4 (Boundary Conditions)

\[ S_0''(x_0) = 0 \Rightarrow a_0 = 0 \]
\[ S_{n-1}''(x_n) = 0 \Rightarrow a_n = 0 \]
(This will yield a natural spline)

\[
\begin{pmatrix}
4 & 1 & 0 \\
1 & 4 & 1 \\
\vdots & \ddots & \ddots \\
0 & \ddots & 4 & 1 \\
& & & 1 & 4 \end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n-2} \\
a_{n-1}
\end{pmatrix}
= \frac{6}{h^2}
\begin{pmatrix}
f_0 - 2f_1 + f_2 \\
f_1 - 2f_2 + f_3 \\
\vdots \\
f_{n-3} - 2f_{n-2} + f_{n-1} \\
f_{n-2} - 2f_{n-1} + f_n
\end{pmatrix}
\]

Coefficient matrix is symmetric, tridiagonal, strictly diagonally dominant, positive definite

Thus, the above system (\( \ast \)) has a solution.
Recall that \( A = (a_{ij}) \) is strictly diagonally dominant if
\[
|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n
\]

Thus, strictly diagonally dominant matrix is invertible.

In practice, coefficients \( a_1, a_2, \ldots, a_{n-1} \) are computed using Gaussian elimination of a tridiagonal matrix.

**Note**: Clamped BCs: \( S'_0 (x_0) = f'(x_0) \)
\[
S'_{n-1} (x_n) = f'(x_n)
\]

Coefficients \( a_0 \) and \( a_1 \) are not zero anymore.

\[
S'_0 (x) = -\frac{a_0}{2h} (x_1 - x)^2 + \frac{a_1}{2h} (x - x_0)^2 - \left( \frac{f_0}{h} - \frac{a_0}{6} \right) + \left( \frac{f_1}{h} - \frac{a_1}{6} \right)
\]
\[ S_0(x_0) = -\frac{a_0}{2} h - \left( \frac{f_0}{h} - \frac{a_0 h}{6} \right) + \left( \frac{f_1}{h} - \frac{a_1 h}{6} \right) = f_0 \]

Similarly, condition \( S_{n-1}(x_n) = f_n \) gives another equation for \( a_0 \) and \( a_1 \).

Equations \((**1)\) and \((**2)\) together with \( n-1 \) unknown variables \( a_0, a_1, \ldots, a_{n-1} \) will form a system of \( n+1 \) unknowns.
In some applications, there are additional injection points in the interior of the interval, which are not at the endpoints due to the boundary conditions. In fact, the natural cubic spline interpolation has injection points at the endpoints.

\[ \forall \eta \left( (x)_x f \right) \frac{\max_{\theta \leq \eta \leq \eta_+} \Theta_2}{\Theta_2} \geq |(x)_x f - (x)f| \]

Note:

\[ u \cdot 0 = 1, \quad \frac{u}{\lambda} = \eta, \quad \eta_+ + \eta = \frac{1}{2}x, \quad \frac{1}{2} \leq x \leq 1 - \frac{1}{2} \frac{\eta_+ + \frac{1}{2}}{\frac{1}{2}} = (x)f \]

\( \Theta_2 \) : natural cubic spline interpolation
Let \( f(x) \) be defined on \([a, b]\), \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) and let \( S(x) \) be a cubic spline interpolant of \( f \) with natural or clamped boundary conditions.

1. \( \left| f(x) - S(x) \right| \leq \frac{5}{384} \max_{a \leq x \leq b} \left| f''(x) \right| \cdot h^4 \)

where \( h = \max_{i} \left| x_{i+1} - x_i \right| \).

2. \( \int_{a}^{b} \left[ S''(x) \right]^2 \, dx \leq \int_{a}^{b} \left[ f''(x) \right]^2 \, dx \)

The first condition says that cubic spline interpolation is 4th order accurate.

Recall

\[
K(x) = \frac{\left| f''(x) \right|}{\left( 1 + [f'(x)]^2 \right)^{3/2}} \approx \left| f''(x) \right|
\]

curvature
\[ \int_a^b [f''(x)]^2 \, dx \] is a crude measure of the total curvature over \([a, b]\).

The second result can be interpreted as optimality property or minimal curvature property. It means if one considers any other interpolant, it will oscillate at least as much as a spline (natural spline: \(S''(x_0) = S''(x_n) = 0\) or clamped spline: \(S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n)\).

**Numerical Integration**

**Newton–Côtes Formulas**

Basic idea

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^{n} c_i \cdot f(x_i)
\]

For now, assume \(x_i = a + i \cdot h\), \(h = \frac{b-a}{n}\)
Trapezoidal Rule

\[ T(h) = h \left( \frac{f(x_0) + f(x_1)}{2} + \ldots + \frac{f(x_{n-1}) + f(x_n)}{2} \right) \]

\[ T(h) = h \left( \frac{1}{2} f(x_0) + f(x_1) + \ldots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right) \]
\[ E_x = \int_0^1 e^{-x^2} \, dx = 0.746824 \ldots \]

<table>
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<th>( h )</th>
<th>( T(h) )</th>
<th>error</th>
<th>error ( /h^2 )</th>
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This implies that the Trapezoidal Rule is 2nd order accurate.

We want to prove this analytically.

Local error analysis (Trapezoidal Rule)