Note.

If two floating point numbers with \( n \) significant digits are subtracted, then the result may have fewer than \( n \) significant digits. This is called loss of precision or loss of significant digits due to cancellation of digits.

\[
\text{Ex}
\]

\[
x = \underline{0.1234}, \quad y = \underline{0.1233}
\]

0.1234 - 0.1233 = 0.0001 = (0.1000)_{10} \cdot 10^{-3}

the result has at most one correct significant digit
Ex (pg. 45) quadratic formula

\[ ax^2 + bx + c = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

0.2 \( x^2 - 47.91x + 6 = 0 \)

\( a = 0.2, \ b = -47.91, \ c = 6 \)

\[ \Rightarrow x_1 = 239.4247, \ x_2 = 0.1253: \text{ computed by Matlab} \]

Now suppose we use 4 digit arithmetic.

\[ x = \frac{47.91 \pm \sqrt{47.91^2 - 4(0.2)6}}{2(0.2)} \]

\[ = \frac{47.91 \pm \sqrt{2290}}{0.4} = \frac{47.91 \pm 47.85}{0.4} \]

\[ \Rightarrow x_1 = \frac{47.91 + 47.85}{0.4} = \frac{95.76}{0.4} = 239.4: \text{ all 4 digits are correct} \]
The problem is due to cancellation of digits since

The remedy is to use a higher precision arithmetic

we subtract two close numbers: 47.91 and 47.85.

Now: 

\[ x = \frac{A - B}{2a} \]

\[ = \frac{A^2 - B^2}{2a(B^2 - Yac)} \]

\[ = \frac{2b + \sqrt{b^2 - 4ac}}{2c} \]

\[ = \frac{2b + \sqrt{b^2 - 4ac}}{2c} \]

\[ = 0.1253 \]

Now, all the digits are correct!
Note: \( n = 4 \)

\[
x = \pm (0. \, d_1 \, d_2 \, d_3 \, d_4)_{10} \cdot 10^e, \quad -M \leq e \leq M
\]

\[
x = 0.000\,125\,3700 \neq 0.000\,1
\]

\[
\text{round correct} \quad f(1)(x) = 0.1254 \cdot 10^{-3}
\]

Other methods to eliminate cancellation of digits:
- Taylor expansion
- Trigonometric identities
- Properties of \( \ln, \exp \) etc.

Ex: Finite difference approximation of a derivative

Recall

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
**Forward Difference**

\[ f'(x) \approx \frac{f(x+h) - f(x)}{h} \]

\[ D_+ f(x) \]

**Question: How large is the error?**

**Taylor series of } f(x) \text{ about } x = a.**

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \]

**Equivalent form,**

\[ x \rightarrow x + h \]

\[ a \rightarrow x \]

\[ \Rightarrow f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \ldots \]
\[ f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!} f''(x) - \ldots \]

Hence, the error is proportional to \( h \). We can write this as

\[ f'(x) = D_t f(x) + O(h) \]

\[ O(h) \approx C \cdot h \quad \text{or} \]

\[ |f'(x) - D_t f(x)| \leq C \cdot h \]

where symbol \( O(h) \) means "of order of \( h \)."

For example, if \( f(x) = e^x \), \( x = 1 \)

\[ f'(x) = e^x, \quad f'(1) = e^1 = 2.71828\ldots : \text{exact value} \]
<table>
<thead>
<tr>
<th>( h )</th>
<th>( D^h \psi )</th>
<th>( \psi'(x) - D^h \psi )</th>
<th>( \frac{\psi'(x) - D^h \psi}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.8588</td>
<td>-0.1406</td>
<td>-1.4056</td>
</tr>
<tr>
<td>0.05</td>
<td>2.7874</td>
<td>-0.0691</td>
<td>-1.3821</td>
</tr>
<tr>
<td>0.025</td>
<td>2.7525</td>
<td>-0.0343</td>
<td>-1.3705</td>
</tr>
<tr>
<td>0.0125</td>
<td>2.7353</td>
<td>-0.0171</td>
<td>-1.3648</td>
</tr>
</tbody>
</table>

\[
\frac{e}{2} = -\frac{1}{2} \psi''(x)
\]
3. For \( h > 10^{-10} \), the error increases as \( h \) is reduced, due to finite precision arithmetic.

2. For \( h < 10^{-10} \), the error is linearly proportional to \( h \).

1. For \( h < 10^{-10} \), the error decreases as \( h \) is reduced, due to the discrete approximation.

\[
\frac{1}{2} f - f^+ A
\]

\[
\frac{2}{3} / 1 = \eta
\]

i.e.,

\[
\text{error log error desired}
\]

\[
\text{error log error vs. log error}
\]

\[
\text{plot log(error)}^{-1}(\eta)
\]

end

end

\[
\text{dip}(\eta)
\]

if \( \text{error} \neq 0 \)
end

for \( j = 1 \) to \( n \)
\[
\text{for } i = 1 \text{ to } n
\]
end