\(\frac{df}{dx} = x^3\)

\[
\int_0^{2h} x^3 \, dx = \frac{x^4}{4} \bigg|_0^{2h} = 4h^4 \quad \frac{1}{4} \cdot \frac{4}{3} \cdot \frac{4^4}{3} \cdot \frac{h^4}{3}
\]

\[
\frac{1}{3} \cdot (4h^4 + \frac{4}{3} \cdot h^3 + \frac{1}{3} \cdot (2h)^3)
\]

\[
= \left(\frac{4}{3} + \frac{8}{3}\right) h^3 = 4h^3
\]

\[
\Rightarrow \int_0^{2h} f(x) \, dx = \frac{1}{3} f(0) + \frac{4}{3} f(h) + \frac{1}{3} f(2h)
\]

ie. Simpson's rule is exact for polynomials of degree \( \leq 3 \) at least.

\(f(x) = x^4\)

\[
\int_0^{2h} x^4 \, dx = \frac{h^5}{5} = \frac{1}{3} \cdot 0 + \frac{4}{3} \cdot h^4 + \frac{1}{3} \cdot (2h)^4
\]

Hence, Simpson's rule is exact for polynomials of degree \( \leq 3 \), ie. degree of precision for Simpson's rule is \( r = 3 \).

Trapezoid and Simpson's rule are examples of Newton-Côtes formulas.
Orthogonal Polynomials

Define, the inner product of two functions on \([-1, 1]\) as

\[ \int_{-1}^{1} f(x)g(x) \, dx = \langle f, g \rangle \]

Properties

1. \( \langle f, f \rangle \geq 0 \) and \( \langle f, f \rangle = 0 \iff f \equiv 0 \)

   \[ \langle f, f \rangle = ||f||^2 \]

   \[ ||f|| = \sqrt{\langle f, f \rangle} : \text{norm of } f \]

2. \( \langle f, \alpha h + g \rangle = \alpha \langle f, h \rangle + \langle f, g \rangle \)

We say that functions \( f \) and \( g \) are orthogonal if \( \langle f, g \rangle = 0 \)
Ex: $\sin \pi x$ and $\cos \pi x$ are orthogonal on $[-1, 1]$

\[
= \int_{-1}^{1} \sin \pi x \cdot \cos \pi x \, dx = \frac{1}{2} \int_{-1}^{1} \sin 2\pi x \, dx = -\frac{1}{2} \cos \pi x \left. \right|_{-1}^{1} = 0
\]

$h(t) = \sin t \cdot \cos t$

$h(t)$: odd

\[\text{odd} \quad \Rightarrow \quad \int_{-1}^{1} f(x) \, dx = 0 \quad \int_{-1}^{1} f(x) \, dx = 2 \int_{0}^{1} f(x) \, dx \]

$f(-x) = f(x)$: $f$ is even

$f(-x) = -f(x)$: $f$ is odd

\[f \quad \text{is even} \quad \Rightarrow \quad \int_{-1}^{1} \text{odd} = 0
\]

\[\int_{-1}^{1} \text{even} = 2 \int_{0}^{1} f(x) \, dx
\]
\[ \langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x \, dx = 0 : \Rightarrow 1 \text{ and } x \text{ are orthogonal on } [-1, 1] \]

\[ \langle 1, x^2 \rangle = \int_{-1}^{1} 1 \cdot x^2 \, dx = 2 \int_{0}^{1} x^2 \, dx = \frac{2}{3} x^3 \bigg|_{0}^{1} = \frac{2}{3} \neq 0 \]

\[ \Rightarrow 1 \text{ and } x^2 \text{ are not orthogonal} \]

**Gram-Schmidt orthogonalization method**

Given a sequence of linearly independent functions \( \{ \psi_0, \psi_1, \psi_2, \ldots \} \), the Gram-Schmidt orthogonalization method produces a sequence \( \{ \psi_0, \psi_1, \psi_2, \ldots \} \) of mutually orthogonal functions.

In particular, given a sequence \( \{ 1, x, x^2, x^3, \ldots \} \) of linearly independent functions, the Gram-Schmidt process gives a sequence of orthogonal
polynomials \{ p_0(x), p_1(x), p_2(x), \ldots \} called Legendre polynomials.

\begin{align*}
p_0 &= 1 \\
p_1 &= x + \alpha_{10} p_0
\end{align*}

We want to find coefficient \( \alpha_{10} \) such that \( p_1 \) and \( p_0 \) are orthogonal.

\[
\langle p_1, p_0 \rangle = \langle x, p_0 \rangle + \alpha_{10} \frac{\langle p_0, p_0 \rangle}{\|p_0\|^2} \Rightarrow \alpha_{10} = -\frac{\langle x, p_0 \rangle}{\|p_0\|^2} \\
\|p_0\|^2 = \langle p_0, p_0 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2
\]

\[
\langle x, p_0 \rangle = \langle x, 1 \rangle = 0
\]

\[
\|p_0\|^2 = \langle p_0, p_0 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2
\]

\[
\Rightarrow \alpha_{10} = 0 \Rightarrow p_1 = x + 0 \cdot p_0 \Rightarrow p_1 = x
\]

\[
p_2 = x^2 + \alpha_{21} p_1 + \alpha_{20} p_0
\]

We want to find \( \alpha_{21}, \alpha_{20} \) in such a way that \( p_2 \) is orthogonal to both \( p_0 \) and \( p_1 \).
\[
\begin{align*}
\langle p_2 | p_0 \rangle &= \langle x^2, p_0 \rangle + \alpha_{21} \langle p_1 | p_0 \rangle + \alpha_{20} \langle p_0 | p_0 \rangle \\
\langle x^2, p_0 \rangle &= \langle x^2, 1 \rangle = \frac{2}{3} \\
\| p_0 \|_2 &= 2 \\
\langle p_2 | p_1 \rangle &= \langle x^2, p_1 \rangle + \alpha_{21} \langle p_1 | p_1 \rangle + \alpha_{20} \langle p_0 | p_1 \rangle \\
\langle x^2, p_1 \rangle &= \int_{-1}^{1} x^2 dx = 0 \\
\| p_1 \|_2^2 &= \langle p_1, p_1 \rangle = \int_{-1}^{1} x^2 dx = 2 \int_{0}^{1} x^2 dx = \frac{2}{3} \\
\therefore \quad p_2 &= x^2 - \frac{1}{3}
\end{align*}
\]
Summary

\[ p_0 = 1 \]
\[ p_1 = x + \alpha_{10} p_0 = x - \frac{\langle x, p_0 \rangle}{\| p_0 \|} p_0 = x \]
\[ p_2 = x^2 + \alpha_{21} p_1 + \alpha_{20} p_0 = x^2 - \frac{\langle x^2, p_1 \rangle}{\| p_1 \|^2} p_1 - \frac{\langle x^2, p_0 \rangle}{\| p_0 \|^2} p_0 = x^2 - \frac{1}{3} \]

Now

\[ p_3 = x^3 - \frac{\langle x^3, p_2 \rangle}{\| p_2 \|^2} p_2 - \frac{\langle x^3, p_1 \rangle}{\| p_1 \|^2} p_1 - \frac{\langle x^3, p_0 \rangle}{\| p_0 \|^2} p_0 \]

\[ \langle x^3, p_i \rangle = \int_{-1}^{1} x^3 \cdot x \, dx = \frac{2}{5} \quad \text{and} \quad \| p_i \|^2 = \frac{2}{3} \]

\[ \Rightarrow p_3 = x^3 - \frac{3}{5} x \]
Note

1. $P_n(x)$ is a Legendre polynomial of degree $n$

2. Any polynomial of degree $\leq n$ can be written

$$q(x) = \sum_{i=0}^{n} c_i P_i(x)$$

Legendre polynomials $\{P_i(x)\}_{i=0}^{n}$ form a basis of the set of polynomials $P_n$ of degree $\leq n$.

Gaussian Quadrature

1. Legendre polynomials $P_n(x)$ have $n$ distinct roots in $(-1, 1)$, say, $x_i$, $i = 1, \ldots, n$

2. There exist coefficients $c_i$, $i = 1, \ldots, n$ such that

$$\int_{-1}^{1} f(x) \, dx \approx \sum_{i=1}^{n} c_i f(x_i) = G_n$$
is exact for polynomials of degree \( \leq 2n-1 \).

\[
\frac{e}{\sqrt{\pi}} \int_0^1 e^{-x^2} \, dx = \left| \frac{t=2x-1}{dt=2dx} \right| = \int_{-1}^1 e^{-\frac{(t+1)^2}{4}} \cdot \frac{dt}{2} = \int_{-1}^1 \frac{1}{2} e^{-\frac{(t+1)^2}{4}} \, dt
\]

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This is much better than with Trapezoidal or Simpson's rules.