Recall the Taylor expansion of \( f(x) \) about \( x = a \):
\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots
\]

Taylor Thm:
\[
f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}
\]

where \( \xi \) is some value between \( a \) and \( x \).
\[
P_n(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n: \text{ Taylor polynomial of degree } n
\]
\[
R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}: \text{ remainder}
\]
Ex \[ f'(x) \approx \frac{f(x+h) - f(x)}{h} = D_f f(x) \]

\[ \text{Error bound} \]

\[ f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2} h^2 \]

\[ h = x - a \quad \text{where } \xi \text{ is between } x \text{ and } x+h \]

\[ \frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(\xi)}{2} \cdot h \]

\[ \text{exact approximation} \]

\[ \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = \left| \frac{h}{2} f''(\xi) \right| \leq \frac{h}{2} M \]

\[ \text{error} \]

where \[ |f''(\xi)| \leq M = \max |f''(\xi)| \]
Define

Suppose \( \lim_{h \to 0} F(h) = L \)

If there exist constants \( p \) and \( C \) such that

\[ |F(h) - L| \leq C \cdot h^p, \quad C > 0 \]

for sufficiently small \( h \), we write this as

\[ F(h) - L = O(h^p) \quad \text{or} \quad \frac{F(h) - L}{h^p} \sim \text{const} \]

The constant \( p \) is called the order of accuracy.

\( C \): asymptotic constant. We also say that \( F(h) \) converges to \( L \) with the rate of convergence \( O(h^p) \).

\[ \text{Ex} \quad \frac{f(x+h) - f(x)}{h} = f'(x) + O(h) \quad \text{w.l.} \quad p=1, \quad C = \frac{M}{2} \]
We wrote previously

\[ \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = \left| \frac{h}{2} f''(\xi) \right| \leq \frac{M}{2} \cdot h \]

where \( M = \max |f''(\xi)| \)

Note as \( h \downarrow \), error also decreases.
As \( h \) decreases by a half, the error decreases approximately also by a half.

\[
\text{Ex} \quad \frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)
\]

Central difference approximation
Proof

\[
f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(\xi_1)\]

\[
f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(\xi_2)\]

\(\xi_1\) is between \(x\) and \(x+h\)

\(\xi_2\) is between \(x\) and \(x-h\)

Subtract.

\[
f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{6} (f'''(\xi_1) + f'''(\xi_2))
\]

\[
\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{12} (f''(\xi_1) + f''(\xi_2))
\]

approximation

exact

error

\[
error = exact - approximation
\]

\[
\left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| = \frac{h^2}{12} \left| f'''(\xi_1) + f'''(\xi_2) \right| \leq
\]
\[
\frac{h^2}{12} \cdot 2 |f''''(\xi)| = \frac{h^2}{6} |f''''(\xi)| \leq \frac{h^2}{6} \cdot M
\]

\[
\therefore \quad \left| \frac{f(x+h) - f(x-k)}{2h} - f'(x) \right| \leq \frac{M}{6} h^2 \quad p=2 \quad c=\frac{M}{6}
\]

where \( M = \max |f''''(\xi)| \)

Note Approximation \( \frac{f(x+h) - f(x-k)}{2h} \) is 2nd order accurate, whereas \( \frac{f(x+h) - f(x)}{h} \) is only 1st order accurate.

Claim \( \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + O(h^2) \)
Recall notation
\[
D_+ f = \frac{f(x+h) - f(x)}{h}, \quad D_- f = \frac{f(x) - f(x-h)}{h}
\]

\[
\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = D_+ D_- f: \text{ can be shown}
\]

Back to root-finding methods

Note Root-finding methods are used to solve usually nonlinear equations.

Thm Intermediate Value Thm

Suppose \( f(x) \) is continuous on \([a, b]\). Let \( K \) be any number between \( f(a) \) and \( f(b) \), i.e.

\( f(a) < K < f(b) \) or \( f(b) < K < f(a) \)
Then there exists a value \( x \in (a, b) \) such that
\[ f(x) = k. \]

Bisection method uses this idea to find a root of
\[ f(x) = 0. \]

Given
\[ f(a) < 0 < f(b) \]
or \[ f(b) < 0 < f(a) \]
then there exists \( x \in (a, b) \) such that \[ f(x) = 0. \]
Idea of bisection method:
check the sign of \( f(\frac{a+b}{2}) \). Shrink the interval
to subinterval that contains the root (values of \( f \)
have opposite sign).

**Fixed-point iteration**

Suppose that \( f(x) = 0 \) is equivalent to \( x = g(x) \).
We say that \( x \) is a fixed point of \( g \), \( x = g(x) \) \( \iff \)
\( x \) is a root of \( f(x) \), i.e. \( f(x) = 0 \).

We define an iterative scheme
\[ x_{n+1} = g(x_n) \quad \text{given } x_0. \]