# 1

\[ 0.001235 = 0.124 \times 10^{-2} \]

\[ 0.000005671 = 0.567 \times 10^{-5} \]

\[ 0.123 \times 10^{-2} \]

\[ 0.416 \times 10^{-2} \]

\[ 0.123 \times 10^{-2} + 0.416 \times 10^{-2} = (0.123 + 0.416) \times 10^{-2} \]

\[ = 0.539 \times 10^{-2} \]

\[ 0.123 \times 10^{-2} + 0.459 \times 10^{-1} = 0.0123 \times 10^{-1} + 0.459 \times 10^{-1} = 0.471 \times 10^{-1} \]

Add mantissas, round to 3 digits

\[
\begin{array}{c}
+0.0123 \\
0.459 \\
\hline
0.4713 \approx 0.471
\end{array}
\]

# 3

(a) composition of \( D_+ \) and \( D_- \)

(b) use Taylor Thm for \( f(x+h) \), \( f(x-h) \). Combine expansions and solve for \( f''(x) \).
\[|x - x_{n+1}| \leq C |x - x_n|^r\]

- \(r\): order of convergence
- \(C\): asymptotic constant

**Note**
1. \(|x - x_n| \leq K |x - x_{n-1}|\): linear convergence
2. \(K \sim |g'(x)|\): we want to choose the iteration function \(g(x)\) in such a way that \(|g'(x)|\) is as small as possible

**Recall**
\[f(x) = x^2 - 3, \quad \alpha = \sqrt{3}\]
\[g_1(x) = x - \frac{x^2 - 3}{2}, \quad g_1'(x) = 1 - \frac{2x}{2} = 1 - x\]
\[|g_1'(x)| = |g_1' (\sqrt{3})| = 1 - \sqrt{3} = 0.73 < 1\]

\[g_2(x) = \frac{3}{x}, \quad g_2'(x) = -\frac{3}{x^2}\]
\[|g_2' (\sqrt{3})| = \frac{3}{(\sqrt{3})^2} = 1\]
How do we determine the order of convergence numerically?

\[ \{ x_n \} \to \alpha \text{ with order } r \text{ if} \]

\[ |d - x_n| \leq C |\alpha - x_{n-1}|^r \]

error at iteration \( n \)

error at iteration \( n-1 \)

Denote \( E_n = |d - x_n| \): abs. error at iteration \( n \).

\[ \Rightarrow E_n \leq C E_{n-1}^r \quad \text{or} \quad E_n \approx C E_{n-1}^r \]

\[ \ln E_n \approx \ln (C E_{n-1}^r) \]

\[ \ln E_n \approx \ln C + r \ln E_{n-1} \]

Assume that \( C \approx 1 \Rightarrow \ln C \approx 0 \)

\[ \Rightarrow \ln E_n \approx r \ln E_{n-1} \Rightarrow \]

\[ r \approx \frac{\ln E_n}{\ln E_{n-1}} \]
In practice, exact value $x$ may not be known.
Can we still use this result to find order $r$ of convergence?

Yes, if we note that

$$x - x_n \approx x_{n+1} - x_n \quad \Rightarrow \quad \frac{\ln |x_{n+1} - x_n|}{\ln |x_n - x_{n-1}|} \approx r$$

**Newton's Method (S2.4)**

$$f(x) = 0$$

Idea: expand $f(x_{n+1})$ around the point $x = x_n$.

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \ldots$$

$$x_n + (x_{n+1} - x_n) \quad \Rightarrow \quad 0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

Then solve for $x_{n+1}$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ given } x_0$$
Slope \quad f'(x_n) = \frac{f(x_n) - 0}{x_n - x_{n+1}} \Rightarrow \text{solve for } x_n \text{ as above}

\text{Ex. } \quad f(x) = x^2 - 3

\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}

\quad f'(x) = 2x

\quad \Rightarrow x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}

\Rightarrow \text{iteration function } g(x) = x - \frac{f(x)}{f'(x)}
<table>
<thead>
<tr>
<th>n</th>
<th>x_n</th>
<th>f(x_n)</th>
<th></th>
<th>d - x_n</th>
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<td></td>
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<td>1.7320509</td>
<td>0.000001</td>
<td></td>
<td>0.000001</td>
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</tbody>
</table>

**Note**

1. \( x_{n+1} = g(x_n) \) where \( g(x) = x - \frac{f(x)}{f'(x)} \)

\( \frac{df}{dx} = f(x) = 0 \), \( f'(x) \neq 0 \) (\( x \) is a simple root of \( f \))

\[ g'(x) = 0, \quad g''(x) \neq 0 \]

\[ g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g'(x) = 1 - \frac{1}{[f'(x)]^2} \left[ f'(x) \cdot f''(x) - f''(x) \cdot f(x) \right] = \]

\[ \frac{\left( f'(x) \right)^2 - \left( \left( f'(x) \right)^2 - f(x) f''(x) \right) \left( f'(x) \right)^2}{\left( f'(x) \right)^2} = \frac{f(x) f''(x)}{\left( f'(x) \right)^2} \]
$$g'(x) = \frac{f(x^0)f''(x)}{(f'(x))^2} = 0$$

One can show that \( g''(x) = \frac{f''(x)}{f'(x)} \neq 0 \)

2. It can be shown if \( x \) is a simple root of \( f(x) \), then
\[ |x - x_0| \leq C |x - x_{n-1}|^2 : \quad \text{2nd order of convergence} \]

If \( x \) is a multiple root with multiplicity \( m \geq 2 \), then
\[ |x - x_n| \leq C |x - x_{n-1}| : \quad \text{1st order of convergence} \]

3. Newton’s method is more expensive than bisection, fixed-point (in general) since we have two function evaluations \( (f(x_n), f'(x_n)) \) per iteration.
The Convergence of Newton's Method

Suppose function \( f \in C^2[a, b] \) (i.e., \( f \) has continuous second derivative) and assume that \( f \) has a simple root \( \lambda \in (a, b) \), i.e., \( f(\lambda) = 0 \), \( f'(\lambda) \neq 0 \). Then Newton's Method converges to \( \lambda \) if \( x_0 \) is chosen sufficiently close to \( \lambda \).