With red-black ordering equations that result from application of 5-point discrete Laplacian can be written as:

\[
\begin{pmatrix}
D_R & H \\
K & D_B
\end{pmatrix}
\begin{pmatrix}
U_R \\
U_B
\end{pmatrix} =
\begin{pmatrix}
b_R \\
b_B
\end{pmatrix}
\]

where $D_R, D_B$ are scalar diagonal matrices.

Then

\[
\begin{align*}
U_R &= -D_R^{-1}Hu_R + D_R^{-1}b_R \\
U_B &= -D_B^{-1}Ku_R + D_B^{-1}b_B
\end{align*}
\]

\[
\begin{align*}
\mathbf{D}_R \cdot \mathbf{U}_R + H \mathbf{U}_B &= \mathbf{b}_R & | D_R^{-1} \\
K \mathbf{U}_R + D_B \cdot \mathbf{U}_B &= \mathbf{b}_B & | D_B^{-1}
\end{align*}
\]

\[
\begin{align*}
\mathbf{U}_R + D_R^{-1}H \mathbf{U}_B &= D_R^{-1} \mathbf{b}_R \\
\mathbf{U}_B + D_B^{-1}K \mathbf{U}_R &= D_B^{-1} \mathbf{b}_B
\end{align*}
\]
All red points can be calculated in parallel using black points. Then all black points can be calculated using red points.

Polynomial approximation (Chapter 5)

A polynomial \( p(x) \) of degree \( n \) has the form

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \quad a_n \neq 0 \]

The highest power, \( n \) in this case, is called the degree of the polynomial.

\[ \text{Ex} \quad p(x) = 1 + x + x^2 \text{ is a polynomial of degree } 2. \]

Note

1. Any polynomial is a continuous function
2. But not all continuous functions are polynomials

\[ \text{Ex} \quad \sin x \]

\[ f(x) = \frac{1}{1 + x^2} \text{ is continuous but it is not a polynomial} \]
Thm (Weierstrass Thm)

Given a continuous function \( f(x) \), \( x \in [a, b] \). For any \( \varepsilon > 0 \), there exists a polynomial \( p(x) \) such that

\[
|f(x) - p(x)| \leq \varepsilon \quad \text{for all} \quad x \in [a, b]
\]

\[
\max_{a \leq x \leq b} |f(x) - p(x)| \leq \varepsilon
\]

Application

\[
\left| \int_{a}^{b} f(x) \, dx - \int_{a}^{b} p(x) \, dx \right| =
\]

\[
= \left| \int_{a}^{b} (f(x) - p(x)) \, dx \right| \leq \int_{a}^{b} |f(x) - p(x)| \, dx \leq \int_{a}^{b} \varepsilon \, dx = \varepsilon (b - a)
\]
Taylor theorem

Let \( f(x) \) be defined on \([-a,b]\) and suppose \( f^{(n+1)}(x) \) is continuous for all \( x \in [-a,b] \).
Then if \( x_0, x \in [-a,b] \), there exists \( \xi = \xi(x) \) between \( x_0 \) and \( x \) such that

\[
f(x) = p_n(x) + r(x)
\]

where

\[
p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \ldots + f^{(n)}(x_0)(x-x_0)^n
\]

is the degree Taylor polynomial at \( p_n(x) \) and \( x_0 \).

\[
r(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{remainder or error}
\]

\[
\frac{\ln x}{x^2} \quad f(x) = \frac{1}{1+25x^2} \quad p_0 = 1
\]

\[
1 = 1 - 25x^2 \quad p_2 = 1 - 25x^2
\]

\[
1 = 1 - 25x^2 + 625x^4 \quad p_4 = 1 - 25x^2 + 625x^4
\]

\[
1 = 1 - 25x^2 + 625x^4 - 15625x^6 \quad p_6 = 1 - 25x^2 + 625x^4 - 15625x^6
\]

\[
Note \quad \int f(x) \, dx \quad \text{is poorly approximated by} \quad \int p_n(x) \, dx
\]
\[ f(x) = \frac{1}{1 + 25x^2} \]

\[ p_n(x) = \sum_{x=0}^{n} (-25x^2)^k = 1 - 25x^2 + 625x^4 - 15625x^6 + \ldots + (-25x^2)^n \]

\[ \frac{1}{1-z} = 1 + z + z^2 + \ldots \quad |z| < 1 \]

\[ z = -25x^2 \Rightarrow \quad 125x^4 < 1 \Rightarrow \quad x^2 < \frac{1}{125} \]

\[ \Rightarrow \quad |x| < \frac{1}{5} = 0.2 \]

Note:
1. If \( |x| < 0.2 \), then \( \lim_{n \to \infty} p_n(x) = f(x) \)
2. If \( |x| \geq 0.2 \), then \( \lim_{n \to \infty} p_n(x) \) doesn't exist

**Polynomial interpolation**

Let \( f \) be a continuous function and \( x_0, x_1, \ldots, x_n \) distinct points.

**Questions**

1. Does there exist a unique polynomial \( p \)