Notes On Directional Derivatives, Gradient, etc.

Tangent Planes:
- to the surface \( z = f(x, y) \) at the point \((x_0, y_0, z_0)\):
  \[
  z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
  \]
- to the surface \( F(x, y, z) = k \) at the point \((x_0, y_0, z_0)\):
  \[
  F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.
  \]
  (The gradient vector is normal to the surface!)

Differentials:
- \( \Delta z \approx dz = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \)
- \( f(x, y) = f(x_0, y_0) + \Delta z \approx f(x_0, y_0) + dz = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \).

Implicit Slopes:
- on the curve \( F(x, y) = 0 \): \( \frac{dy}{dx} = \frac{-F_x}{F_y} \)
- on the surface \( F(x, y, z) = 0 \): \( \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} \).

Directional Derivatives:
- If \( u \) is a unit vector then \( D_u f = u \cdot \nabla f \).

2nd Derivative Test: If \((a, b)\) is a critical point and \( D = (f_{xx})(f_{yy}) - (f_{xy})^2 \) then
  - \( D(a, b) > 0 \) and \( f_{xx}(a, b) > 0 \) \( \implies \) relative minimum
  - \( D(a, b) > 0 \) and \( f_{xx}(a, b) < 0 \) \( \implies \) relative maximum
  - \( D(a, b) < 0 \) \( \implies \) saddle point

Method of Lagrange Multipliers: To find the candidate points for extreme values of \( f(x, y) \) on the curve \( g(x, y) = k \) solve:
\[
\begin{cases}
\nabla f = \lambda \nabla g \\
g(x, y) = k
\end{cases}
\]