Math 275 (Nielsen),
Exam 2A (100 pts)

Instructions: Work neatly and show all necessary work. Answers without supporting calculations will not receive credit. Clearly indicate your final answer.

1. (8 pts) True or False:
   - T F If $h_x(a,b) = 0$ and $h_y(a,b) = 0$ then the tangent plane to the surface $z = h(x,y)$ at $(a,b, h(a,b))$ will be horizontal.
   - T F If $h_x(a,b) = 0$ and $h_y(a,b) = 0$ then $h(x,y)$ has a local maximum or local minimum at $(a,b)$. (Could be a saddle point!)
   - T F The gradient vector $\nabla f(a,b,c)$ is tangent to the surface $f(x,y,z) = f(a,b,c)$ at the point $(a,b,c)$. (Normal, not tangent!) 
   - T F The maximum value of $f(x,y,z)$ on the surface $g(x,y,z) = 0$ will occur at a point where $\nabla f$ is parallel to $\nabla g$.

2. (8 pts) Show that the following limit does not exist.

   \[
   \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}
   \]

   Along y-axis: \[
   \lim_{(0,y) \to (0,0)} \frac{0}{0^2 + y^2} = 0
   \]

   Along line $y = x$: \[
   \lim_{(x,x) \to (0,0)} \frac{x^2}{x^2 + x^2} = \lim_{(x,y) \to (0,0)} \frac{x^2}{2x^2} = \frac{1}{2}
   \]

   Since these limits don't agree, the limit as $(x,y) \to (0,0)$ doesn't exist!

3. (12 pts) Let $f(x,y) = y \sin(xy^2 + 1)$. Find the following partial derivatives.
   - (a) $f_x(x,y) = y \cos(xy^2 + 1)$, $f_y(x,y) = y^3 \cos(xy^2 + 1)$
   - (b) $f_y(x,y) = \sin(xy^2 + 1) + y \cos(xy^2 + 1) \cdot 2xy$
     \[
     = \sin(xy^2 + 1) + 2xy^2 \cos(xy^2 + 1)
     \]
   - (c) $f_{xy}(x,y) = 3y^2 \cos(xy^2 + 1) - y^3 \sin(xy^2 + 1) \cdot 2xy$
     \[
     = 3y^2 \cos(xy^2 + 1) - 2xy^4 \sin(xy^2 + 1)
     \]
4. (10 pts) Find the equation of the tangent plane to the graph of \( f(x, y) = x^2 y^2 + 2xy - 5x \) at \((x, y) = (1, 2)\).

\[
\begin{align*}
f_x &= 2xy^2 + 2y - 5, \quad f_x(1, 2) = 8 + 4 - 5 = 7 \\
f_y &= 2x^2 y + 2x, \quad f_y(1, 2) = 4 + 2 = 6 \\
f(1, 2) &= 4 + 4 - 5 = 3
\end{align*}
\]

\[
z - 3 = 7(x - 1) + 6(y - 2)
\]

5. (12 pts) Suppose \( z = x^2 y^2 - 5xy + 3x - 7y \) where \( x = g(s, t) \) and \( y = h(s, t) \) with \( g(3, -2) = 5, h(3, -2) = -1, g_t(3, -2) = 5, g_s(3, -2) = 0, h_s(3, -2) = -4, \) and \( h_t(3, -2) = 1 \). Use the chain rule to evaluate \( \frac{\partial z}{\partial s} \) at \((s, t) = (3, -2)\).

\[
\begin{align*}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\
&= (2xy^2 - 5y + 3) \cdot g_s + (2x^2 y - 5x - 7) \cdot h_s \\
\Rightarrow \frac{\partial z}{\partial s} \bigg|_{(s,t)=(3,-2)} &= (10 + 5 + 3) \cdot 5 + (-50 - 25 - 7) \cdot (-4) \\
&= 90 + 328 = 418
\end{align*}
\]

6. (8 pts) Find \( \frac{dy}{dx} \) for the curve \( \sqrt{x^2 + y^3} = 6xy \).

\[
\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-x}{\sqrt{x^2 + y^3}} + 6y \\
F(x, y) = 0
\]

\[
\frac{-x}{\sqrt{x^2 + y^3}} + 6y = \frac{-\frac{x}{\sqrt{x^2 + y^3}} + 6y}{\frac{2y^2}{2\sqrt{x^2 + y^3}} - 6x}
\]

7. (10 pts) Find the rate of change in the function \( f(x, y) = x e^{-y^2} \) at the point \((4, 2)\) in the direction towards the point \((1, 3)\).

\[
\nabla f = \left< e^{-y^2} + xe^{-y^2}, -2y e^{-y^2} \right>
\]

\[
\nabla f(4, 2) = \left< e^0 + 4e^0, -16e^0 \right> = \left< 5, -16 \right>
\]

\[
\frac{\partial u}{\partial x} f(1, 3) = \left< 5, -16 \right> \cdot \left< -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right> = \frac{-31}{\sqrt{10}}
\]
8. (12 pts) The function \( f(x, y) = xy^2 + x^3 - 3x^2 - y^2 + 3 \) has four critical points. Find them.

\[
\frac{f_x}{\partial x} = 0 \\
y^2 + 3x^2 - 6x = 0
\]

\( y = 0 \Rightarrow 3x^2 - 6x = 0 \)
\[ 3x(x - 2) = 0 \]
\[ x = 0, 2 \]
\( \Rightarrow (0, 0), (2, 0) \)

\( x = 1 \Rightarrow y^2 + 3 - 6 = 0 \)
\[ y = \pm \sqrt{3} \]
\( \Rightarrow (1, \sqrt{3}), (1, -\sqrt{3}) \)

\[ \frac{f_y}{\partial y} = 0 \\
2xy - 2y = 0 \]
\[ 2y(x - 1) = 0 \]
\[ y = 0 \text{ or } x = 1 \]
\[
\{ (0, 0), (2, 0), (1, \sqrt{3}), (1, -\sqrt{3}) \}
\]

9. (8 pts) The function \( g(x, y) = x^2 + 2y^2 - xy^2 \) has critical points at \((2, 2), (2, -2), \) and \((0, 0)\). Determine if each is a local maximum, a local minimum, or a saddle point.

\[ g_x = 2x - y^2 \]
\[ g_y = 4y - 2xy \]
\[ D = g_{xx} g_{yy} - (g_{xy})^2 = (2)(-4) - (2y)^2 \]
\[ = 4(2 - x) - 4y^2 \]

\[ D(2, 2) < 0 \Rightarrow \text{saddle} \]
\[ D(2, -2) < 0 \Rightarrow \text{saddle} \]
\[ D(0, 0) > 0, \quad f_{xx} = 2 > 0 \Rightarrow \text{minimum} \]

10. (12 pts) Use Lagrange multipliers to find the maximum value of the function \( f(x, y, z) = xyz \) on the surface \( x^2 + 2y^2 + 3z^2 = 6 \).

\[ \nabla f = \lambda \nabla g, \quad \langle yz, xz, xy \rangle = \lambda \langle 2x, 4y, 6z \rangle \]
\[ yz = 2xz \Rightarrow \lambda = \frac{yz}{2x} \]
\[ xz = 4zy \Rightarrow \lambda = \frac{xz}{4y} \]
\[ xy = 6yz \Rightarrow \lambda = \frac{xy}{6z} \]
\[ \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{6z} \]
\[ x^2 + 2y^2 + 3z^2 = 6 \]
\[ 2y^2 + 2y^2 + 2y^2 = 6 \]
\[ \Rightarrow y^2 = 1, \quad y = \pm 1, \quad x = \pm \sqrt{2}, \quad z = \pm \sqrt{3} \]
\[ \Rightarrow \text{Max Value: } f(\sqrt{2}, 1, \sqrt{3}) = \sqrt{2} \cdot 1 \cdot \sqrt{3} = \sqrt{6} \cdot \sqrt{3} = \frac{2}{\sqrt{3}} \]