Construction of Rational Surfaces in Projective Fourspace

(joint work with Kristian Ranestad)

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1. Introduction

$\mathbb{P}^4$:= projective fourspace over $\mathbb{C}$.

Theorem (-, Ranestad, 2004).
There exist five different families of smooth rational surface in $\mathbb{P}^4$ with degree 12 and sectional genus 13.

Remark.
These surfaces are all isomorphic to $\mathbb{P}^2$ blown up in 21 points.

Linear systems

The embedding linear systems of these surfaces are of the following types:

(i) $(12; 4^1, 3^{12}, 2^0, 1^8)$

(ii) $(12; 4^2, 3^9, 2^3, 1^7)$

(iii) $(12; 4^3, 3^6, 2^6, 1^6)$

(iv) $(12; 4^4, 3^3, 2^9, 1^5)$

(v) $(12; 4^5, 3^0, 2^{12}, 1^4)$

A theorem of Severi

$X :=$ smooth surface in $\mathbb{P}^n$.

$\mathrm{Sec}(X) :=$ secant variety to X in $\mathbb{P}^n$.

Theorem (Severi, 1901).
$X :=$ smooth nondegenerate surface in $\mathbb{P}^5$. Then:

$$\mathrm{Sec}(X) \neq \mathbb{P}^5 \Leftrightarrow X \text{ is the Veronese surface.}$$

Double point formula

- X:= smooth surface in $\mathbb{P}^4$.
- H:= its hyperplane class.
- K:= its canonical divisor.
- $d := H^2 = \deg(X)$.
- π:=$\frac{1}{2}H.(H+K)+1$= its sectional genus.
- χ:= Euler-Poincaré characteristic.

$$d^2 - 10d - 5H.K - 12K^2 + 12\chi = 0.$$

Finiteness result of Ellingsrud and Peskine

Theorem (Ellingsrud, Peskine, 1989).

$\exists\, d_0 \in \mathbb{N}$ such that for every smooth surface $X \subset \mathbb{P}^4$ of degree d the following holds:

$$d > d_0 \Rightarrow X \text{ is of general type.}$$

Remark.
The theorem implies that there are only a finite number of families of nongeneral-type surfaces in $\mathbb{P}^4$.

Goal.

Classify the smooth nongeneral type surfaces in $\mathbb{P}^4$.

Problem 1.

Find the true d_0.

Problem 2.

Classify the smooth surfaces in $\mathbb{P}^4$ of small degree.

Known families of smooth nongeneral-type surfaces in $\mathbb{P}^4$ (∼1995) can be found in

W. Decker and S.Popescu: *On surfaces in* $\mathbb{P}^4$ *and* 3*-folds in* $\mathbb{P}^5$, London Math. Soc. Lecture Note Ser., **208**, (1995), 69–100

List of nongeneral-type surfaces found after 1996

d	rational	ruled	Enriques	elliptic
8	0	1 [ADS]	0	0
11	3 [S] +1 [BEL]	0	1 [S]	0
12	5 [AR2, AS] +1	0	0	1 [AR1]

References

[ADS] H. Abo, W. Decker, N. Sasakura: *An elliptic conic bundle in* $\mathbb{P}^4$ *arising from a stable rank-*3 *vector bundle*, Math. Z. **229** (1998), 725–741.

[AR1] H. Abo, K. Ranestad: *Irregular elliptic surfaces of degree* 12 *in projective fourspace*, Math. Nachr. **278** (2005), 511–524.

[AR2] H. Abo, K. Ranestad: *Construction of rational surfaces in projective fourspace*, preprint.

[AS] H. Abo, F.-O. Schreyer, *Exterior algebra methods for the construction of rational surface in* $\mathbb{P}^4$, preprint.

[BEL] H.-C. Bothmer, C. Erdenberger, K. Ludwig; *A new family of rational surfaces in* $\mathbb{P}^4$, available at from arXiv server as `math.AG/0404492`

[S] F.-O. Schreyer, *Small fields in vonstructive algebraic geometry*, Lecture Notes in Pure and Appl. Math., **179** (1994) 221–228.

Remarks.

- The classification of the smooth nongeneral type surfaces has been completed up to degree 10.
- A partial classification in degree 11 has been given (Sorin Popescu, 1993).

Construction

Take the following steps:

Step 1. Prove (or disprove) the existence of a smooth surface in $\mathbb{P}^4$ with given invariants such as degree and sectional genus.

Step 2. Determine where the surface stands in the Enriques classification table (use **Adjunction theory**).

Construction methods.

- Linear systems on abstract surfaces.
- Liaison.
- Eagon-Northcott complex method (Decker, Ein and Schreyer).

Adjunction theory

Neglecting some well-known exceptions, we have

Theorem (Sommese, Van de Ven, 1987).

The adjoint linear system $|H+K|$ defines a birational morphism

$$\Phi_{|H+K|} : X \longrightarrow \mathbb{P}^{\pi - p_a - 1}$$

onto a smooth surface X_1, which blows down precisely all (-1)-curves on X.

The Eagon-Northcott complex method

$\mathcal{E}$:= vector bundle on $\mathbb{P}^4$ with $\text{rank}(\mathcal{E}) = r$.

$\mathcal{F}$:= vector bundle on $\mathbb{P}^4$ with $\text{rank}(\mathcal{F}) = r + 1$.

φ:= morphism from $\mathcal{E}$ to $\mathcal{F}$ such that

$$X := \left\{ \; p \in \mathbb{P}^4 \; \middle| \; \text{rank}(\varphi) < r \; \right\}$$

has codimension 2, then X is locally Cohen-Macaulay.

Conversely, every locally Cohen-Macaulay subscheme of codimension 2 in $\mathbb{P}^4$ arises in this way.

Beilinson's theorem

Theorem (Beilinson, 1978).

For any sheaf $\mathcal{F}$ on $\mathbb{P}^n$, there is a complex $\mathcal{K}^\cdot$ with

$$\mathcal{K}^i = \bigoplus H^{i-j}(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(j)) \otimes \Omega^{-j}(-j)$$

such that

$$H^i(\mathcal{K}^\cdot) = \begin{cases} \mathcal{F} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition.

We call $\mathcal{K}^\cdot$ a **Beilinson monad** for $\mathcal{F}$.

2. Construction of rational surfaces in $\mathbb{P}^4$

V:= 4-dimensional vector with basis $\{e_i\}_{0\leq i\leq 4}$.

W:= its dual with dual basis $\{x_i\}_{0\leq i\leq 4}$

X := smooth surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with

$$d = 12,\ \pi = 13 \text{ and } p_g = q = 0.$$

$\mathcal{I}_X$:= ideal sheaf of X.

Beilinson's theorem tells us that $\mathcal{I}_X(4)$ is obtained via

$$0 \longrightarrow 4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O}_{\mathbb{P}^4} \longrightarrow 0.$$

Let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_2 & B_1 \end{pmatrix}$.

For fixed A_1 and B_1, the matrix equation

$$B \circ A = B_2 \circ A_1 + B_1 \circ A_2 = 0$$

gives rise to a homogeneous system of 120 linear equations with 140 unknowns.

$F :=$ coefficient matrix of the system of linear equations.

rank(F)	Existence	Linear System
120	$\times$	$-$
119	?	?
118	?	?
117	$\bigcirc$	$(12; 4^5, 3^0, 2^{12}, 1^4)$
116	$\bigcirc$	$(12; 4^4, 3^3, 2^9, 1^5)$
115	$\bigcirc$	$(12; 4^3, 3^6, 2^6, 1^6)$
114	$\bigcirc$	$(12; 4^2, 3^9, 2^3, 1^7)$
113	$\bigcirc$	$(12; 4^1, 3^{12}, 2^0, 1^8)$

How to find A_1 and B_1

Fix a "general" $B_1 \in \mathrm{Hom}(2\Omega^1(1), 3\mathcal{O})$. For example,

$$B_1 = \begin{pmatrix} e_0 & e_1 \\ e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}.$$

$S_B :=$ locus in $\mathbb{P}(V)$, where

$$B_1 : \mathrm{H}^1(\mathbb{P}^4, \mathcal{I}_X(3)) \to \mathrm{H}^1(\mathbb{P}^4, \mathcal{I}_X(4))$$

is not injective.

For an $A_1 \in \mathrm{Hom}(4\Omega^3(3), 2\Omega^2(2))$,

C_A:= locus in $\mathbb{P}(V)$, where

$$A_1 : \mathrm{H}^2(\mathbb{P}^4, \mathcal{I}_X(1)) \to \mathrm{H}^2(\mathbb{P}^4, \mathcal{I}_X(2))$$

is not surjective.

For a given $N \in \{113, \ldots, 117\}$, find an A_1 such that

(a) C_A is a rational normal curve in $\mathbb{P}(V)$.

(b) $\mathrm{rank}(F) = N$.

$\mathfrak{F}$:= family of rational normal curves in $\mathbb{P}(V)$.

$\cup$

$\mathfrak{F}_N$:= subfamily of rational normal curves in $\mathbb{P}(V)$

satisfying (b).

$c := \operatorname{codim}(\mathfrak{F}, \mathfrak{F}_N)$.

$\mathbb{F}_p$:= finite field with p elements.

Performing a random search, we can expect to find a point of $\mathfrak{F}_N(\mathbb{F}_p)$ from $\mathfrak{F}(\mathbb{F}_p)$ at a late of $(1 : p^c)$.

Question. Is $\operatorname{codim}(\mathfrak{F}, \mathfrak{F}_N) = (120 - N)^2$?

V_A:= column space of A_1.

V_B:= row space of B_1.

Every column of A_1 and every row of B_1 have rank 2, so they define elements in $G(2, V) \subset \mathbb{P}(\bigwedge^2 V)$.

The corresponding maps of $\mathbb{P}(V_A)$ and $\mathbb{P}(V_B)$ into $G(2, V)$ are the double embeddings.

Z_A:= image of $\mathbb{P}(V_A) \to$ Veronese 3-fold.

Z_B:= image of $\mathbb{P}(V_B) \to$ Veronese surface.

Lemma 1.

The intersection of Z_A and Z_B consists of at most 6 points.

Lemma 2.

If Z_A and Z_B intersect at k points, then

$$\operatorname{rank}(F) \leq 120 - k.$$

Corollary. $\operatorname{codim}(\mathfrak{F}, \mathfrak{F}_N) \leq 120 - N.$

Lift to characteristic zero

Lemma (Schreyer, 1996).

Let $A_1 \in \mathrm{Hom}(4\Omega^3(3), 2\Omega^2(2))$ satisfying (a) and (b). If $\mathfrak{F}_N$ has codimension $120 - N$ at the point x corresponding to A_1. Then:

$$\exists \text{ a number field } \mathbb{L} \text{ and } \exists \text{ a prime } \mathfrak{p} \text{ in } \mathbb{L}$$

such that the residue field $\mathcal{O}_{\mathbb{L},\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ is in $\mathbb{F}_p$. Furthermore, if $X/\mathbb{F}_p$ corresponding to x is smooth, then $X/\mathbb{L}$ corresponding to the generic point of $\mathrm{Spec}(\mathbb{L}) \subseteq \mathrm{Spec}(\mathcal{O}_{\mathbb{L},\mathfrak{p}})$ is also smooth.

For a given $A_1 \in \mathrm{Hom}(4\Omega^3(3), 2\Omega^2(2))$, each family has dimension 38.

$$(*) \quad 38 \geq (140 - N - 20) - 1 + 18 + \dim(\mathfrak{F}_N),$$

where

18 = dim. of the family of rational cubic scrolls;

$140 - N$ = dimension of the solution space and

20 = dimension of the "trivial" solution space.

From (*) it follows that $\dim(\mathfrak{F}_N) = N - 99$. So we have

$$\begin{aligned} \operatorname{codim}(\mathfrak{F}, \mathfrak{F}_N) &\geq 21 - (N - 99) \\ &= 120 - N, \end{aligned}$$

where 21 = dimension of the family of rational normal curves. By Corollary,

$$\operatorname{codim}(\mathfrak{F}, \mathfrak{F}_N) = 120 - N.$$

Problems

Problem 1.

Does there exist a smooth rational surface of degree 13?

Problem 2.

Find a geometric construction of each family.