Problem Set 2

Problem 1. *a) Let* $I = (x^2 - 1)$ *be an ideal in* $\mathbb{R}[x]$. *Is* I *maximal? Why or why not?*

b) Let $I = (x^2 + 1)$ *be an ideal in* $\mathbb{R}[x]$. *Is* I *maximal? Why or why not?*

c) Let $I = (x^2 + 1)$ *be an ideal in* $\mathbb{C}[x]$. *Is* I *maximal? Why or why not?*

Problem 2. *a) Let* I *be an ideal in a ring* R. *Suppose* $f^n \in I$ *and* $g^m \in I$. *Show that* $(f + g)^{n+m} \in I$.

b) Let $J = \{f \in R | f^t \in I \text{ for some } t > 0\}$. *Show that* J *is an ideal.*

Remark 3. *The ideal* J *defined in the problem above is a radical ideal and is the smallest radical ideal which contains* I. *Hence* $J = rad(I)$. *Many books define the radical in this way i.e. they define* $rad(I) = \{f \in R | f^t \in I \text{ for some } t > 0\}$. *It is equivalent to the definition given in the handout entitled "Some Useful Definitions and Facts from Commutative Algebra". It is interesting to note that it was only fairly recently (1992) that an algorithm was published giving a symbolic computation of the radical of an ideal.*

Problem 4. *Let* $I = (x^2, xy)$ *be an ideal.*

a) Show that $rad(I) = (x)$.

b) Show that I *is not primary.*

c) Write I *as the intersection of two primary ideals.*

Problem 5. *Let* $I = (x^2 - 1, y^2 - 4) \subset \mathbb{R}[x, y]$. *Write* I *as the intersection of 4 maximal ideals.*

Problem 6. *Let* $I = (x^2, xy^2, y^3)$. *Find the dimension of* $\mathbb{C}[x, y]/I$ *as a vector space over* $\mathbb{C}$ *and produce a basis for this vector space.*

Problem 7. *Let* $I = (y^2 - x^2, y^2 + x^2)$. *Find the dimension of* $\mathbb{C}[x, y]/I$ *as a vector space over* $\mathbb{C}$ *and produce a basis for this vector space.*

Problem 8. *Let* k *be any field and let* $F \in k[x]$ *be a polynomial of degree 5. Find a basis for* $k[x]/(F)$ *as a vector space over* k.

Problem 9. *In definition 19 and 20 of the handout "Some Useful Definitions and Facts from Commutative Algebra", you can find the terms: Module Finite and Ring Finite. Let* R *be a ring and let* $S = R[x]$.

a) Is S *module finite over* R*?*

b) IS S *ring finite over* R*?*

Problem 10. *Suppose* R *and* S *are rings. Show that if* S *is module finite over* R *then* S *is ring finite over* R.

The following problem is from the book Ideals, Varieties, and Algorithms by Cox, Little and O'Shea.

Problem 11. *In engineering, it is an important problem to be able to build up a complicated curve or surface (such as the outside of a car) out of simple pieces (because simple pieces are easier to construct). To make the pieces fit together in a smooth manner, it is required that the pieces look very similar, locally, at the points of intersection. For curves, this means that they have the same tangent lines at the points of intersection (sometimes the sharing of higher order tangents may also be required). Bézier cubic curves provide a simple model that accomplishes this feat. Given 4 points in the plane* $(x_0, y_0), \dots (x_3, y_3)$*, a Bézier cubic curve is a cubic curve that starts at* (x_0, y_0) *with tangent line passing through* (x_1, y_1) *and ends at* (x_3, y_3) *with tangent line passing through* (x_2, y_2)*. The Bézier cubic is produced by letting* t *vary from 0 to 1 in the expression*

$$x = (1-t)^3 x_0 + 3t(1-t)^2 x_1 + 3t^2(1-t)x_2 + t^3 x_3$$

$$y = (1-t)^3 y_0 + 3t(1-t)^2 y_1 + 3t^2(1-t)y_2 + t^3 y_3.$$

a) Show that (x_0, y_0) *(resp.* (x_3, y_3)*) lie on the curve at times* $t = 0$ *(resp.* $t = 1$*).*

b) Show that the tangent line to the curve at time $t = 0$ *(resp.* $t = 1$*) passes through the points* (x_1, y_1) *(resp.* (x_2, y_2)*).*

Remark 12. *It can also be shown that the Bézier curve stays inside the convex hull of the four points* $(x_0, y_0), \dots (x_3, y_3)$*.*

The next few problems are from Fulton's book on "Algebraic Curves".

Problem 13. *Let* k *be ANY field. Recall that* $k[x]$ *is a UFD. Recall that a polynomial* $F \in k[x]$ *is irreducible if it cannot be factored as* $F = GH$ *with the degree of* G *and* H *both positive.*

a) Prove that there are an infinite number of irreducible polynomials in $k[x]$*. (Hint: Think of Euclid's proof that there are an infinite number of primes).*

b) Use this fact to show that if k *is an algebraically closed field then* k *has an infinite number of elements. (Hint: What are the irreducible polynomials in* $k[x]$ *if* k *is algebraically closed?).*

Problem 14. *Let* k *be a field. Let* $F \in k[x_1, x_2, \dots, x_n]$*. Let* $a_1, a_2, \dots, a_n$ *be elements from* k*.*

a) Show that F *can be written as a* k*-linear combination of elements of the form* $(x_1 - a_1)^{m_1}(x_2 - a_2)^{m_2} \dots (x_n - a_n)^{m_n}$ *(each* $m_i \geq 0$*) by giving an algorithm for writing* F *in this manner. If you prefer lots of notation, this problem is asking you to show that* F *can be written as*

$$F = \sum_{(i_1, i_2, \dots, i_n) \geq (0, 0, \dots, 0)} C_{(i_1, i_2, \dots, i_n)}(x_1 - a_1)^{i_1}(x_2 - a_2)^{i_2} \dots (x_n - a_n)^{i_n} \text{ with } C_{(i_1, i_2, \dots, i_n)} \in k.$$

b) Suppose $F(a_1, a_2, \dots, a_n) = 0$*, then show that* $F = \sum_{i=1}^{n}(x_i - a_i)G_i$ *for some* $G_1, G_2, \dots, G_n \in k[x_1, x_2, \dots, x_n]$*.*