Problem Set 3

Problem 1. *a) Let* $I = (x^4 - 1) \subseteq \mathbb{R}[x]$*. Find* $I(V(I))$*.*
b) Let $I = (x^5 - 1) \subseteq \mathbb{R}[x]$*. Find* $I(V(I))$*.*
c) Let $I = (x^n - 1) \subseteq \mathbb{R}[x]$*. Find* $I(V(I))$*.*

Problem 2. *Give an example of 2 ideals* I *and* J *in* $\mathbb{C}[x, y]$ *such that* $I \neq J$ *but* $V(I) = V(J)$*.*

Problem 3. *Show that the closed affine varieties in* $\mathbb{A}^1_k$ *are the finite subsets of* $\mathbb{A}^1_k$ *and all of* $\mathbb{A}^1_k$*.*

I thank George Raptis for posing the following question:

Problem 4. *Show that if* k *is a finite field, then EVERY subset of* $\mathbb{A}^n_k$ *is a closed affine variety. (Hence we can conclude that the Zariski topology on* $\mathbb{A}^n_k$ *is the same as the discrete topology on* $\mathbb{A}^n_k$*!)*

These next few problems are from Fulton's "Algebraic Curves":

Problem 5. *Let* k *be a field and let* I *be an ideal with* $I \subseteq k[x_1, x_2, \ldots, x_n]$*.*
a) Show that $V(I) = V(rad(I))$*.*
b) Show that $rad(I) \subseteq I(V(I))$*.*

Problem 6. *Let* V, W *be closed affine varieties. Show that* $V = W$ *if and only if* $I(V) = I(W)$*.*

Problem 7. *a) Let* V *be a closed algebraic variety in* $\mathbb{A}^n$ *and let* P *be a point in* $\mathbb{A}^n$ *which is not in* V*. Show that there exists a polynomial* $F \in k[x_1, x_2, \ldots, x_n]$ *such that* $F(Q) = 0$ *for every point in* V *but* $F(P) = 1$*.*

b) Let $\{P_1, P_2, \ldots, P_n\}$ *be a finite set of points in* $\mathbb{A}^n$*. Show that there exist polynomials* $\{F_1, F_2, \ldots, F_n\} \in k[x_1, x_2, \ldots, x_n]$ *such that* $F_i(P_j) = 0$ *whenever* $i \neq j$ *but* $F_i(P_i) = 1$*.*

c) Let V *be a closed algebraic variety in* $\mathbb{A}^n$ *and let* P_1, P_2 *be two points in* $\mathbb{A}^n$ *which are not in* V*. Show that there is a polynomial* $F \in I(V)$ *such that* $F(P_1) \neq 0$ *and* $F(P_2) \neq 0$*.*

The problems on this page will encourage you to start the process of learning Macaulay II (or some other equivalent Computer Algebra system).

We will soon learn the following theorem:

Theorem 8. *(Weak Nullstellensatz) Let k be an algebraically closed field. If $I \subseteq k[x_1, x_2, \ldots, x_n]$ is a proper ideal then $V(I) \neq \emptyset$.*

Let $F, G \in \mathbb{C}[x]$. If F and G do not share a common root then $V((F,G)) = \emptyset$. As a consequence of the Weak Nullstellensatz, this implies that (F,G) is not a proper ideal. In other words $(F,G) = (1)$. This implies that $1 \in (F,G)$. The following general question is very natural:

Question 9. *If I is an ideal in $k[x_1, x_2, \ldots, x_n]$ and if $P \in k[x_1, x_2, \ldots, x_n]$, how can we determine if $P \in I$?*

Grobner bases will give a method for answering this question. The computer algebra system "Macaulay II" has various Grobner basis algorithms incorporated into its commands. In fact, the ideal membership question is a command built into Macaulay II. With this in mind, you have two possible methods for answering the following problem, either the method of resultants with the polynomials F and G or testing whether $1 \in (F,G)$. With either method, Macaulay II will make your work easier and I suggest you use it. Hiro will be available to help answer questions on getting started with Macaulay II.

Problem 10. *Let $F = x^5 + 4x^3 + 6x^2 + x + 2$ and let $G = 4x^3 + 7x^2 + 2x + 1$. Do F and G share a root?*

Let $V = \{P_1, P_2, \ldots, P_t\} \subseteq \mathbb{A}^n$ be a finite collection of points. There are two basic methods to determine $I(V) \subseteq k[x_1, x_2, \ldots, x_n]$. The first method uses linear algebra and involves building up $I(V)$ step by step. We will go over this method but, unfortunately, it is difficult to determine when you have completed the process. I.e. it is difficult to determine when you have a set of elements which generate $I(V)$. A second method is to determine $I(P_i)$ for each $P_i \in V$ and then compute $J = I(P_1) \cap I(P_2) \cap \cdots \cap I(P_t)$. Since J is precisely the set of functions which vanish at P_1 and P_2 and $\ldots$ and P_t, we can conclude that $J = I(V)$. This brings up the following natural question:

Question 11. *Given two ideals I and J, can you compute the ideal $I \cap J$?*

Grobner bases again let you solve this problem. Macaulay II has a built in function that allows you to compute the intersection of 2 ideals. You should use Macaulay II to help you solve the following problem:

Problem 12. *Let $P_1 = (1,2,3), P_2 = (1,3,7), P_3 = (2,3,5)$. Let $V = \{P_1, P_2, P_3\}$. Find $I(V)$.*