# Problem Set 4

Let $V$ be a closed affine variety in $\mathbb{A}_k^n$ and let $W$ be a closed affine variety in $\mathbb{A}_k^m$. The **product** of $V$ and $W$ will be defined as

$$V \times W = \{(a_1, \ldots, a_n, b_1, \ldots, b_m) | (a_1, \ldots, a_n) \in V \text{ and } (b_1, \ldots, b_m) \in W\}.$$

**Problem 1.** *Let $V$ be a closed affine variety in $\mathbb{A}_k^n$ and let $W$ be a closed affine variety in $\mathbb{A}_k^m$. Show that $V \times W$ is a close affine variety in $\mathbb{A}_k^{n+m}$.*

**Problem 2.** *Consider the closed affine variety $V(Y^2 - X) \subseteq \mathbb{A}_{\mathbb{C}}^2$. Show that $V(Y^2 - X)$ is irreducible. (Hint: Show that $I(V(Y^2 - X))$ is prime).*

**Problem 3.** *Let $I = (Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3) \subseteq \mathbb{C}[X, Y]$.*
*a) Decompose $V(I)$ into three irreducible components.*
*b) Use the decomposition to construct $I(V(I))$. (Hint: Macaulay 2 can help).*

**Problem 4.** *Let $I = (x^2 + y^2 - 1, x^2 - z^2 - 1) \subseteq \mathbb{C}[x, y, z]$. Decompose $V(I)$ into 2 irreducible components.*

**Problem 5.** *Let $X = \{(0, 0)\} \in \mathbb{A}_{\mathbb{R}}^2$.*
*a) Find a single polynomial $F \in \mathbb{R}[x, y]$ such that $X = V(F)$.*
*b) Is $I(X) = (F)$?*

**Problem 6.** *Let $I_1 = (x + 1, y + 1), I_2 = (x, y - 2), I_3 = (x - y)$ be three ideals in $\mathbb{R}[x, y]$. Let $V_1 = V(I_1), V_2 = V(I_2), V_3 = V(I_3)$. It is easy to check that $V_1, V_2, V_3$ are each irreducible. Let $V = V_1 \cup V_2 \cup V_3$. Show that there exists an $F \in \mathbb{R}[x, y]$ such that $V = V(F)$.*

*The previous 2 problems illustrate that unusual behavior can occur when working over a field which is not algebraically closed. One can extend the idea of the solution of problem 6 to show that any closed affine variety in $\mathbb{A}_{\mathbb{R}}^2$ is $V(F)$ for some element $F \in \mathbb{R}[x, y]$. This is definitely not true if we work in $\mathbb{A}_k^2$ when $k$ is algebraically closed. For most of the course, we will work over an algebraically closed field and this will simplify things considerably.*

**Problem 7.** *Let $J_1 = (x), J_2 = (x^2, y), J_3 = (x^2, xy, y^2)$. It can be shown that $J_1, J_2, J_3$ are all primary ideals. Compute $I_1 = J_1 \cap J_2$ and $I_2 = J_1 \cap J_3$.*

**Definition 8.** *An ideal, $I$, is said to be* **irreducible** *if it cannot be written as $I = J_1 \cap J_2$ with $J_1, J_2$ ideals satisfying $I \neq J_1, I \neq J_2$.*

**Lemma 9.** *In a Noetherian ring, irreducible ideals are primary.*

*Recall that a closed affine variety, $V$, is irreducible if it cannot be written as $V = V_1 \cup V_2$ with $V_1, V_2$ closed affine varieties satisfying $V \neq V_1, V \neq V_2$. Problem 7 demonstrates that it is possible to have $I$ reducible but $V(I)$ irreducible. In general, if $k$ is an algebraically closed field and if $I \subseteq k[x_1, x_2, \ldots, x_n]$, we have*

*a) $I$ reducible $\not\Longrightarrow$ $V(I)$ reducible (or $V(I)$ irreducible $\not\Longrightarrow$ $I$ irreducible).*
*b) $I$ irreducible $\Longrightarrow$ $V(I)$ irreducible (or $V(I)$ reducible $\Longrightarrow$ $I$ reducible).*
*c) $V(I)$ irreducible $\Longleftrightarrow$ $I(V(I))$ irreducible.*

*What causes the problem in a) is the possibility of "embedded primes" (embedded primes are defined in the handout "Some Useful Definitions and Facts from Commutative Algebra"). If an ideal is a radical ideal, then it will not have any embedded primes. Any ideal can be written as the intersection of primary ideals. A radical ideal can be written as the intersection of prime ideals. In fact, an ideal is a radical ideal if and only if it can be written as the intersection of prime ideals.*

**Definition 10.** *Let $R$ be a ring and let $I, J$ be ideals in $R$. The* **ideal quotient** *of $I$ by $J$ is defined as $I : J = \{f \in R | fJ \subseteq I\}$.*

**Problem 11.** *a) Show that $I : J$ is an ideal.*
*b) Let $F \in R$. Show that $(I \cap J) : F = (I : F) \cap (J : F)$.*

**Problem 12.** *Compute the following ideal quotients (each ideal is in $k[x, y]$).*
*a) $(x^2, xy) : x$*
*b) $(x^2, xy) : y$*
*c) $(x^2, xy) : x^3 + 3xy$*
*d) $(x^3, x^2y) : x$*
*e) $(x^3, x^2y) : y$*
*f) $(x, y)^3 : x$*
*g) $(x, y)^3 : (x, y)$*

*The saturation of $I$ with respect to $J$ is written $I : J^\infty$ and is defined as $I : J^\infty = \cup_{i \geq 1} I : J^i$. By the Noetherian property, $I : J^\infty = I : J^N$ for $N$ sufficiently large. An alternate way to compute $I : J^\infty$ is to define $I_1 = I : J$ and define $I_{d+1} = I_d : J$. This produces a sequence of ideals, $I_1 \subseteq I_2 \subseteq \ldots$, which will stabilize at some point (i.e. for $N$ sufficiently large, $I_N = I_{N+1} = I_{N+2} = \ldots$.)*

**Problem 13.** *Show that $I : J^{d+1} \subseteq I : J^d$ for any $d$.*

**Problem 14.** *Compute the following:*
*a) $(x^3, x^2y) : x^\infty$*
*b) $(x, y)^3 : x^\infty$*