# Problem Set 4

The following problem is meant to illustrate the idea in the proof of Hilbert's Nullstellensatz.

**Problem 1.** *Let $k$ be an algebraically closed field. Let $I = (A^2, B^2) \subseteq k[A, B]$. It is clear that $AB \in I(V(I))$ but that $AB \notin I$. By Hilbert's Nullstellensatz, some power of $AB$ is in $I$ (it is clear, in fact, that $(AB)^2 \in I$ but let's approach this in the spirit of the proof of the Nullstellensatz). Let $I' = (A^2, B^2, 1 - (AB)Y) \subseteq k[A, B][Y]$. Then $V(I') = 0$ so $1 \in I'$.*

*a) Find $F, G, H \in k[A, B][Y]$ such that $1 = FA^2 + GB^2 + H(1 - (AB)Y)$.*

*b) Plug $Y = \frac{1}{AB}$ into the expression $1 = FA^2 + GB^2 + H(1 - (AB)Y)$.*

*c) Clear denominators and write $(AB)^n$ as an element of $I$ for some $n$.*

For the next problem, let $R = k[x_1, x_2, \ldots, x_n]$ and let $I$ be an ideal in $R$ which is generated by monomials. The problem is meant to convince you that, with some work, you can systematically compute $dim_k(R/I)$.

**Problem 2.** *a) Find $dim_k(k[x, y]/(x^4, x^2y^3, y^4))$.*

*b) Find $dim_k(k[x, y, z]/(x^4, x^2y^3, y^4, xyz, z^2))$.*

*c) For a given $I$, what is a quick way to determine if $dim_k(R/I) < \infty$?*

*d) Explain how to determine $dim_k(R/I)$ (when it is finite).*

**Problem 3.** *a) Find $dim_k(k[x, y]/(x^2 - 1, y^2 - 4))$.*

*b) Find a basis for $k[x, y]/(x^2 - 1, y^2 - 4)$ as a vector space over $k$.*

**Problem 4.** *Let $F = \sum_{i=0}^{n} a_i x^i$ and let $G = \sum_{i=0}^{m} b_i y^i$. Find $dim_k(k[x, y]/(F, G))$.*

**Problem 5.** *Let $k$ be an algebraically closed field. Show that each of the following are irreducible in $k[x, y]$.*

*a) $F = y^2 - xy^2 - x^2 - x^3$.*
*b) $G = y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$.*
*c) $H = y^2 - x(x - 1)(x - \lambda)$ for any $\lambda \in k$.*