Problem Set 6

The picture below lies in $\mathbb{A}^2_{\mathbb{R}}$. Each of the 3 line segments in the picture represents a steel bar of length 1. Let $A = (0,0), B = (0,2), C = (x_1, y_1)$ and $D = (x_2, y_2)$. Points A and B cannot move but points C and D can move. The steel bars are connected with hinges. Point M is exactly in the middle of the bar. It turns out that as the three bars move into every allowable position, the point M sweeps out a curve. This curve is an irreducible affine variety given as $V(F)$ for some polynomial $F \in \mathbb{R}[X,Y]$. In the problems following the picture, you will compute F using Macaulay 2 and elimination theory.

Problem 1. *Give a rough sketch of the curve traced out by M.*

Now we will construct an ideal, I, in $\mathbb{R}[X, Y, x_1, x_2, y_1, y_2]$ which represents all allowable configurations and the corresponding positions of M. Each bar yields a constraint on the variables x_1, x_2, y_1, y_2, this yields 3 quadratic polynomials. Let $M = (X, Y)$ and write down the coordinates of M in terms of x_1, x_2, y_1, y_2, this yields 2 linear polynomials. Let I be the ideal generated by the three quadratic polynomials and the two linear polynomials.

Problem 2. *Write out the equations for I.*

We would like to know all of the allowable values of X and Y. This corresponds to computing $J = I \cap \mathbb{R}[X,Y]$. If you carry out this computation in Macaulay 2 you will obtain an ideal with a single generator, F. F is the equation of the curve.

Problem 3. *What is F?*

The next problem should look familiar. Don't let the problem give you a headache, it really is easy once you understand what it is asking.

Problem 4. *Let k be an algebraically closed field. Let $F = y^n + a_{n-1}(x)y^{n-1} + a_{n-2}(x)y^{n-2} + \cdots + a_1(x)y + a_0(x)$ be an irreducible polynomial in $k[x,y]$. Let $V = V(F)$.*

a) Show that the natural homomorphism of $k[x]$ to $\Gamma(V) = k[x,y]/(F)$ is injective (so $k[x]$ is a subring of $\Gamma(V)$).

b) Show that the residues $\overline{1}, \overline{y}, \ldots, \overline{y^{n-1}}$ generate $\Gamma(V)$ as a module over $k[x]$.

The next problem is a small review problem:

Problem 5. *Let $F(x) = x^4 + 2x^3 + 3x^2 + 2x + 2 \in \mathbb{C}[x]$. Given that $i - 1$ is a root, find the primary decomposition of F.*

A **form** *is a polynomial in which every term has the same total degree (like $x^2yz + w^4 + y^2z^2 + w^2xy$).*

Problem 6. *Show that any factor of a form is a form.*

Problem 7. *Show that the set of all affine mappings of $\mathbb{A}^2$ with the operation of composition form a group, I.e. Show the following*
a) There is an identity element.
b) The composition of two affine maps is an affine map.
c) Each affine map has an inverse.
d) The associative property holds.

Problem 8. *a) Show that $f = 3xy + x^3 + y^3 \in \mathbb{C}[x,y]$ is irreducible.*

b) Use part a) to show that $g = x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$ is irreducible. (Hint: irreduciblity is preserved under affine mappings.)

Problem 9. *a) Let A, B, C be three non collinear points in $\mathbb{A}^2$. Show that there exists a unique affine map which takes $(0,0)$ to A, $(1,0)$ to B and $(0,1)$ to C.*

b) Use part a) to show that if A, B, C and D, E, F are sets of non-collinear points, then there is a unique affine map which takes A to D, B to E and C to F.

Problem 10. *Let $A_1, A_2, \ldots A_{n+1}$ and $B_1, B_2, \ldots B_{n+1}$ be sets of non-cohyperplaner points in $\mathbb{A}^n$. Show that there is a unique affine map which take A_i to B_i. (You may assume $Af(n)$ is a group.)*