Problem Set 8

For this entire problem set, k is an algebraically closed field.

Problem 1. *Let $n \geq r$. The projection map $\pi : \mathbb{A}^n \to \mathbb{A}^r$ is given by $\pi(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_r)$. Show that π is a polynomial map.*

Problem 2. *Let $V = V(x^2 - y, x^3 - z)$. Let $\phi : \mathbb{A}^1 \to V$ be defined by $\phi(t) = (t, t^2, t^3)$. Show that ϕ is an isomorphism by constructing a map $\psi : V \to \mathbb{A}^1$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are each identity maps.*

Problem 3. *Let $V = V(y^2 - x^3)$. Let $\phi : \mathbb{A}^1 \to V$ be defined by $\phi(t) = (t^2, t^3)$.*

a) Show that ϕ is a $1 - 1$ and onto polynomial map.

b) Show that ϕ is NOT an isomorphism.

Problem 4. *Let $V = V(y^2 - x^2(x + 1))$. Let $\phi : \mathbb{A}^1 \to V$ be defined by $\phi(t) = (t^2 - 1, t(t^2 - 1))$. Show that ϕ is onto and that ϕ is 1-1 (with the exception that $\phi(1) = \phi(-1) = (0, 0)$).*

Problem 5. *Let $I = (x^2 - y^3, y^2 - z^3)$. Let $\alpha : k[x, y, z] \to k[t]$ be defined by $\alpha(x) = t^9, \alpha(y) = t^6, \alpha(z) = t^4$.*

a) Show that every element of $k[x, y, z]/I$ has a representative of the form $A + Bx + Cy + Dxy$ where $A, B, C, D \in k[z]$.

b) Let $F = A + Bx + Cy + Dxy$ with $A, B, C, D \in k[z]$. Show that $\alpha(F) = 0$ implies that $F = 0$. (Hint: Compare powers of t).

c) Show that $ker(\alpha) = I$.

d) Let $V = V(x^2 - y^3, y^2 - z^3)$. What is the polynomial map $\phi : \mathbb{A}^1 \to V$ that induces α?

e) Show that ϕ is 1-1 and onto (so V is irreducible and $I(V)$ is prime).

f) Show that ϕ is not an isomorphism.

Let P_1, P_2 be two points in $\mathbb{A}^2$. Let L_1, L_2 be two distinct lines through P_1. Let K_1, K_2 be two distinct lines through P_2.

Problem 6. *Show that there is an affine change of coordinates T such that $T(P_1) = P_2$, $T(L_1) = K_1$ and $T(L_2) = K_2$. (Hint: Problem 9 from Set 6 might be helpful)*

Problem 7. *Let $V = V(y^2 - x^2(x+1))$. Let $\overline{x}, \overline{y}$ be the equivalence class of x and y in $\Gamma(V)$. Let $f = \overline{y}/\overline{x} \in k(V)$.*

a) Find the pole set of f.

b) Find the pole set of f^2.

Problem 8. *Let $\phi : \mathbb{A}^1 \to \mathbb{A}^4$ be defined by $t \to (t+1, t^2+2, t^3+3, t^4+4)$. This induces a homomorphism $\widetilde{\phi} : k[w,x,y,z] \to k[t]$.*

a) Is ϕ an isomorphism? Why or why not?

b) Find the kernel of $\widetilde{\phi}$. (Macaulay 2 will be very helpful for this part).

c) Is $ker(\widetilde{\phi})$ a prime ideal?

Problem 9. *Let $\phi : \mathbb{A}^2 \to \mathbb{A}^5$ be defined by $(x,y) \to (x, y, x^2, xy, y^2)$. This induces a map $\widetilde{\phi} : k[A,B,C,D,E] \to k[x,y]$.*

a) Find $ker(\widetilde{\phi})$.

b) Is ϕ an isomorphism?

Problem 10. *Let V be a variety in $\mathbb{A}^n$. Let $F \in \Gamma(V)$. Define $Gr(F) = \{(a_1, a_2, \ldots, a_n, a_{n+1}) \in \mathbb{A}^{n+1} | (a_1, a_2, \ldots, a_n) \in V \text{ and } a_{n+1} = F(a_1, a_2, \ldots, a_n)\}$. $Gr(F)$ is called the graph of F.*

a) Show that $Gr(F)$ is an affine variety.

b) Show that $Gr(F)$ is isomorphic to V.