Problem Set 9

For this entire problem set, $R = k[x_1, x_2, \ldots, x_n]$ with k a field.

Problem 1. *Define a map* $\phi : \mathbb{A}^1 \to \mathbb{A}^4$ *by* $t \to (t, t^2, t^3, t^4)$. *This induces a map* $\widetilde{\phi} : k[A, B, C, D] \to k[t]$.

a) Find $ker(\widetilde{\phi})$.

b) Compute $J = ker(\widetilde{\phi}) \cap k[A, B, C]$.

c) How does this compare with the ideal of the twisted cubic?

d) Compute $I = ker(\widetilde{\phi}) \cap k[B, C]$.

e) How does this compare with the ideal in Problem 3 on Set 8?

f) Is $I = J \cap k[B, C]$*?*

Problem 2. *Let* $I = (x^2 - y^2, xy - 1)$ *be an ideal in* $k[x, y]$.

a) Compute a Gröbner basis for I *with respect to the lex order.*

b) Find a reduced Gröbner basis for I *with respect to the lex order.*

c) Compute $in_{lex}(I)$.

d) Compute $dim_k(k[x, y]/I)$.

e) Compute $I \cap k[y]$.

Problem 3. *Let* $I = (x^2 - y^2, xy - 1)$ *be an ideal in* $k[x, y]$.

a) Compute a Gröbner basis for I *with respect to the hlex order.*

b) Find a reduced Gröbner basis for I *with respect to the hlex order.*

c) Compute $in_{hlex}(I)$.

d) Use c) to compute $dim_k(k[x, y]/I)$.

Problem 4. *Let* $I = (x^2 - y^2, xy - 1)$ *be an ideal in* $k[x, y]$. *Compute a Gröbner basis for* I *with respect to the rlex order.*

Problem 5. *Let* $I = (x^4 - y^4, y^7 - 1)$ *be an ideal in* $k[x, y]$.

a) Compute a Gröbner basis for I *with respect to the hlex order.*

b) Find $dim_k(k[x, y]/I)$.

Problem 6. *Let* $a, b, c \in \mathbb{Z}^+$ *with* $a \geq b$. *Let* $\lambda \in k$. *Use hlex order to find* $dim_k(k[x, y]/(x^a - y^b, y^c - \lambda))$.

Lemma 7. *Let* $I = (x^{a_1}, x^{a_2}, \ldots, x^{a_t}) \in k[x_1, x_2, \ldots x_n]$ *be a monomial ideal. Let* x^a *be a monomial. Then*

$$I : x^a = \left(\frac{x^{a_1}}{GCD(x^{a_1}, x^a)}, \frac{x^{a_2}}{GCD(x^{a_2}, x^a)}, \ldots, \frac{x^{a_t}}{GCD(x^{a_t}, x^a)}\right).$$

Lemma 8. *Let* $R = k[x_1, x_2, \ldots, x_n]$. *Let* $f \in R$. *There is a short exact sequence of the form*

$$0 \to R/(I : f) \to R/I \to R/(I, f) \to 0.$$

Recall that in a short exact sequence of vector spaces $0 \to A \to B \to C \to 0$, $dim(B) = dim(A) + dim(C)$.

Problem 9. *Compute* $\dim_k(k[x, y, z]/(x^3, y^4, z^5, xyz))$ *using Lemma 7 and 8.*

Problem 10. *Let* $I = (y - x^2, z - x^3)$ *be the ideal of the twisted cubic in* $k[x, y, z]$. *Order the degree one monomials by* $y > x > z$. *Let* $F = y^2 z$.

a) Compute a Gröbner basis for I *with respect to lex order (using the given ordering of degree one monomials).*

b) Using the division algorithm, find the remainder upon dividing F *by the elements in the Gröbner basis of* I.