# Problem Set 1

The next several problems are concerned with Sylvester's matrix and the Sylvester determinant. Assume coefficients are drawn from some fixed algebraically closed field.

**Problem 1.** *Let* $F_1 = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *and* $F_2 = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$*. Show that* $F_1$ *and* $F_2$ *share a root if and only if there exists* $G, H$ *with* $deg(G) = m-1$*,* $deg(H) = n-1$ *and* $GF_1 = HF_2$*.*

**Problem 2.** *Let* $S = \{F_1, xF_1, \ldots, x^{m-1}F_1, F_2, xF_2, \ldots, x^{n-1}F_2\}$ *and let* $P_t$ *denote the vector space of all polynomials of degree less than or equal to* $t$*. Use the result in Problem 1 to show that* $S$ *is a linearly dependent set of vectors in* $P_{m+n-1}$ *if and only if* $F_1$ *and* $F_2$ *share a root.*

**Problem 3.** *By writing each element of* $S$ *in terms of the basis* $\{1, x, x^2, \ldots, x^{m+n-1}\}$*, produce an* $(m+n)\times(m+n)$ *matrix* $M$*. Then use the previous problems to conclude that* $det(M) = 0$ *if and only if* $F_1, F_2$ *share a root.*

**Problem 4.** *Devise an algorithm to determine how many roots* $F_1$ *and* $F_2$ *share.*

**Problem 5.** *Suppose* $F_1 = ax^2 + bx + c$ *and* $F_2 = dx^2 + ex + f$ *share a root. Can you conclude that* $G_1 = cx^2 + bx + a$ *and* $G_2 = fx^2 + ex + d$ *share a root?*

**Problem 6.** *Let* $M = \begin{bmatrix} a & b & c & 0 & 0 \\ 0 & a & b & c & 0 \\ 0 & 0 & a & b & c \\ d & e & f & g & 0 \\ 0 & d & e & f & g \end{bmatrix}$*. Assuming* $a, d \neq 0$*, show that* $det(M) = 0$ *if and only if there is an element in the null space of* $M$ *of the form* $\begin{bmatrix} r^4 \\ r^3 \\ r^2 \\ r \\ 1 \end{bmatrix}$ *for some* $r$*.*

**Problem 7.** *Use Sylvester's determinant to find the discriminant of* $F = ax^3 + bx^2 + cx + d$ *(assume* $a \neq 0$*).*

*The next two problems and the remark following them are about Vandermonde matrices:*

**Problem 8.** *It is easy to check that* $\det \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} = (b-a)$. *Now consider the matrix* $M = \begin{bmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{bmatrix}$. *Show that* $\det(M) = (b-a)F(x)$ *where* $F(x)$ *is a monic degree 2 polynomial which factors as* $F(x) = (x-a)(x-b)$. *Conclude that* $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

**Problem 9.** *As a follow up to the previous problem, show that*

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & x \\ a^2 & b^2 & c^2 & x^2 \\ a^3 & b^3 & c^3 & x^3 \end{bmatrix} = (b-a)(c-a)(c-b)F(x)$$

*where* $F(x)$ *is a monic degree 3 polynomial which factors as* $F(x) = (x-a)(x-b)(x-c)$.

**Remark 10.** *Let* $M = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^x & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}$. *The previous 2 problems demonstrate how you could inductively prove that* $\det(M) = \Pi_{i>j}(x_i - x_j)$. $M$ *is called a Vandermonde matrix. Note that if values are plugged in for* $x_1, x_2, \dots, x_n$ *then the matrix is nonsingular if and only if all the values are distinct. As a consequence, any subset of the columns are linearly independent. This may be useful in the next problem (depending on how you choose to solve it).*

**Problem 11.** *If* $M$ *is an* $n \times n$ *matrix, define the corank of* $M$ *to be* $n - \mathrm{rank}(M)$. *Show that if 2 polynomials share* $t$ *distinct roots, then the corank of the associated Sylvester matrix is at least* $t$.