Problem Set 10

For this entire problem set, $R = k[x_1, x_2, \ldots, x_n]$ with k a field.

Recall that if $F, g_1, g_2, \ldots, g_t \in R$, then there is an expression of the form:

$$G = F - \sum_{i=1}^{t} a_i g_i \text{ where } a_i \in R \text{ and}$$

1) None of the monomials of G are in $(in_>(g_1), in_>(g_2), \ldots, in_>(g_t))$.

2) $in_>(F) \geq in_>(a_i g_i)$ for every i.

Also recall that such a G is called a remainder upon dividing F by the elements $g_1, g_2, \ldots, g_t$ and that such remainders are not usually unique.

Problem 1. *Show that if $G = \{g_1, g_2, \ldots, g_t\}$ is a Gröbner basis and $F \in R$ then the remainder upon dividing F by the elements of G is unique.*

Problem 2. *Let $I = (x^3y - x^2, x^2 - xy)$. Compute $I : x^2$ using the algorithm for ideal quotients. (You can use Macaulay 2, just don't use the ideal quotient command).*

Problem 3. *Let $F \in R$ and let I be an ideal in R. Let $J = (I, 1 - Fy)$ be an ideal in $k[x_1, \ldots, x_n, y]$. Show that $I : F^\infty = J \cap k[x_1, \ldots, x_n]$.*

Problem 4. *Show that $rad(I \cap J) = rad(I) \cap rad(J)$.*

The next problem is a bit silly once you discover the answer. Still it illustrates how you need to be a bit careful.

Problem 5. *Give an example to show that $rad(IJ) \neq rad(I)rad(J)$.*

The following is an unsolved problem. Maybe one of you will solve it some day (or even this weekend). I will state it purely as an algebra problem. Geometrically, the problem is asking if the monomial quartic curve in $\mathbb{P}^3$ is a "set theoretic complete intersection". A more general open problem: "Is every irreducible curve in $\mathbb{P}^3$ a set theoretic complete intersection?". It is surprising that the answer is not even known for the monomial quartic.

Problem 6. *Let $I = (bc - ad, c^3 - bd^2, b^3 - a^2c, ac^2 - b^2d)$ be an ideal in $S = \mathbb{C}[a, b, c, d]$. Do there exist two homogeneous polynomials $F, G \in S$ such that $rad(F, G) = I$?*