Problem Set 11

For this entire problem set, $R = k[x_1, x_2, \ldots, x_n]$ with k a field.

Let $F \in k[x_1, x_2, \ldots, x_n]$. F can be written as a sum $F = \sum_i F_i$ where each F_i is homogeneous of degree i. Recall that an ideal, I, is homogeneous if $F \in I \implies F_i \in I$ for each i.

Problem 1. *Prove that an ideal is homogeneous* $\iff$ *the ideal is generated by a finite set of homogeneous forms.*

Problem 2. *Let* $R = k[x, y, z]$. *Let* F *be an irreducible form of degree* t. *Let* $\Gamma = \Gamma(V(F))$. *Consider the exact sequence* $0 \to R \overset{\times F}{\to} R \to \Gamma \to 0$. *Let* $\Gamma_d = \{$*Forms of degree* d *in* $\Gamma\}$.

a) Show that Γ_d *is a finite dimensional* k*-vector space.*

b) Find $\dim_k(\Gamma_d)$.

Recall that to $\mathbb{P}^n$ *we can associate a dual space* $\mathbb{P}^{n*}$. *Each point in* $\mathbb{P}^n$ *corresponds to a hyperplane in* $\mathbb{P}^{n*}$ *by* $[a_1 : a_2 : \cdots : a_n] \to V(\sum_i a_i y_i)$. *To a linear space,* $L \in \mathbb{P}^n$ *we can associate a linear space* $L^* \in \mathbb{P}^{n*}$ *by* $L^* = \cap_{P \in L} P^*$.

Problem 3. *Let* $L = V(w + y + z, w + x + 2y + z) \subseteq \mathbb{P}^3$. *Find* $I(L^*) \subseteq \mathbb{P}^{3*}$.

Problem 4. *Let* $I = (w^2 - x, w^3 - y, w^4 - z) \subseteq k[w, x, y, z]$. *Let* $J = (w^2 - xh, w^3 - yh^2, w^4 - zh^3)$.

a) Find the ideal of the projective closure of $V(I)$ *by computing* $L = J : h^\infty$ *where* h *is the homogenizing variable.*

b) Find the ideal of the projective closure of $V(I)$ *by computing a Gröbner basis of* I *with respect to glex then homogenize the elements in the Gröbner basis to get an ideal* L'.

c) Check that $L = L'$.

Problem 5. *Let* $I = (x^2 - y, x^3 - z)$*. Let* I^h *be the homogenization of* I *with respect to* h*. Let* $J = (x^2 - yh, x^3 - zh^2)$*.*

a) Check that $J = I^h \cap (J : I^h)$*.*

b) Find $rad(J : I^h)$*.*

c) I^h *is a radical ideal and* $rad(J) = rad(I^h) \cap rad(J : I^h)$*. Use this to compute* $rad(J)$*.*

Problem 6. *a) The Veronese surface,* $V \subseteq \mathbb{P}^5$*, is the projective closure of the image of the map* $\phi : \mathbb{A}^2 \to \mathbb{A}^5$ *given by* $\phi(x, y) = (x, y, x^2, xy, y^2)$*. Find the ideal of the projective closure.*

b) We can also obtain the Veronese surface by considering the map $\phi : \mathbb{P}^2 \to \mathbb{P}^5$ *given by* $\phi([x : y : z]) \to [x^2 : xy : xz : y^2 : yz : z^2]$*. Find the ideal of the Veronese using this map.*

Let t *be one less than the number of degree* s *monomials in* $s + 1$ *variables. The d-uple embedding of* $\mathbb{P}^s$ *into* $\mathbb{P}^t$ *is given by* $\phi([x_0 : x_1 : \cdots : x_s] = [x^{\alpha_0} : x^{\alpha_2} : \cdots : x^{\alpha_t}]$ *where* $\alpha_0, \alpha_1, \ldots, \alpha_t$ *is a basis for space of monomials of degree* d *in* $s + 1$ *variables. The Veronese surface from the previous problem is the 2-uple embedding of* $\mathbb{P}^2$ *into* $\mathbb{P}^5$*.*

Problem 7. *Find the ideal of the 3-uple embedding of* $\mathbb{P}^2$ *into* $\mathbb{P}^9$*.*